math 321 Hw 11 Solutions

Dp 4.2

(optional)

65. We must show adj A is invertible, $(\text{adj } A)^{-1} = \frac{1}{\det A} A$, and $\text{adj } (A^{-1}) = \frac{1}{\det A} A$.

adj A: We will first show if A is invertible, then $B = kA$ $(k \neq 0)$ is invertible.
We will then use this fact to prove that $\text{adj } A = (\det A)A^{-1}$ is invertible.
Since A is invertible implies $\det A \neq 0$, if $B = kA$, then $\det B = k^n \det A \neq 0$.
So, Theorem 4.6 implies that B is invertible.
Now by Theorem 4.12, $A^{-1} = \frac{1}{\det A} \text{adj } A$, so $\text{adj } A = (\det A)A^{-1}$.
Since A^{-1} is invertible, $\det A \neq 0$. Therefore, $\text{adj } A = (\det A)A^{-1}$ is invertible.
Q: How might we convey the above argument (loosely) in words?
A: Since adj A is a (nonzero) scalar multiple of an invertible matrix, adj A is invertible.

$(\text{adj } A)^{-1}$: As in Section 3.3, we prove X is the inverse of adj A by showing $(\text{adj } A)X = I$.
We will show $(\text{adj } A)(\frac{1}{\det A} A) = I$ which will imply $(\text{adj } A)^{-1} = \frac{1}{\det A} A$.
$(\text{adj } A)(\frac{1}{\det A} A) = ((\det A)A^{-1})(\frac{1}{\det A} A) = (\det A \frac{1}{\det A})(A^{-1}A) = A^{-1}A = I$

$\text{adj } (A^{-1})$: Note that Theorem 4.12 asserts $A^{-1} = \frac{1}{\det A} \text{adj } A$, so $\text{adj } A = (\det A)A^{-1}$.
So $(A^{-1})^{-1} = A = \frac{1}{\det(A^{-1})} \text{adj } (A^{-1})$, so $\text{adj } (A^{-1}) = (\det(A^{-1}))A$.
Now recall that $\det(A^{-1}) = \frac{1}{\det A}$, so $\text{adj } (A^{-1}) = (\det(A^{-1}))A = \frac{1}{\det A} A$ as claimed.

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(optional)

70. We show that *block* form can be used to compute determinants with certain restrictions.

(a) Below is an example of P, Q, R, and S all square such that $\det A \neq (\det P)(\det S) - (\det Q)(\det R)$.

Let $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $R = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$.

Therefore, $A = \begin{bmatrix} P & Q \\ R & S \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$.

So, $\det A = \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{vmatrix} = 0 \neq 1 = (1)(1) - (0)(0) = (\det P)(\det S) - (\det Q)(\det R)$.

(b) Given the proof of Exercise 69, it is clear the result holds for *lower* block form as well. That is, if A, P, and S are square and $A = \begin{bmatrix} P & O \\ Q & S \end{bmatrix}$, then $\det A = (\det P)(\det S)$.

So, since $B = \begin{bmatrix} P^{-1} & O \\ -RP^{-1} & I \end{bmatrix}$, Exercise 69 implies $\det B = (\det P^{-1})(\det I) = \det P^{-1}$.

Since P is invertible $\det B = \det P^{-1} \neq 0$. So, B is invertible and $A = B^{-1}BA$.

Furthermore, $\det B^{-1} \overset{\text{Thm 4.9}}{=} \frac{1}{\det B} = \frac{1}{\det(P^{-1})} \overset{\text{Thm 4.9}}{=} \frac{1}{1/\det P} = \det P$.

Also since $B = \begin{bmatrix} P^{-1} & O \\ -RP^{-1} & I \end{bmatrix}$ and $A = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$, we have the following:

$$BA = \begin{bmatrix} P^{-1} & O \\ -RP^{-1} & I \end{bmatrix} \begin{bmatrix} P & Q \\ R & S \end{bmatrix} = \begin{bmatrix} I & P^{-1}Q \\ -RP^{-1}P + R & -RP^{-1}Q + S \end{bmatrix} = \begin{bmatrix} I & P^{-1}Q \\ O & S - RP^{-1}Q \end{bmatrix}.$$

So Exercise 69 implies $\det(BA) = (\det I)(\det(S - RP^{-1}Q)) = \det(S - RP^{-1}Q)$.

Now, $\det A = \det(B^{-1}(BA)) \overset{\text{Thm 4.8}}{=} (\det(B^{-1}))(\det(BA)) = (\det P)(\det(S - RP^{-1}Q))$.

(c) We use Theorem 4.8 to prove $\det A = \det(PS - RQ)$ provided $PR = RP$. So:

$$\det A = (\det P)(\det(S - RP^{-1}Q)) \overset{\text{Thm 4.8}}{=} \det(P(S - RP^{-1}Q))$$
$$= \det(PS - (PR)P^{-1}Q) \overset{PR=RP}{=} \det(PS - (RP)P^{-1}Q) \overset{\text{assoc}}{=} \det(PS - R(PP^{-1})Q)$$
$$\overset{P \text{ invertible}}{=} \det(PS - RQ)$$ as we were to show.

HW 11 Solutions

math 227

(1)

DP 4.1

7. As in Example 4.2, we show null$(A-3I) \neq \mathbf{0}$ then compute null$(A-3I)$ to find $\mathbf{x}$.

Since $A\mathbf{x} = 3\mathbf{x}$ implies $(A-3I)\mathbf{x} = \mathbf{0}$, we have:

$$A - 3I = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}$$

Since the columns of $A-3I$ are clearly linearly dependent (because $\mathbf{a}_2 = -2\mathbf{a}_1$), the Fundamental Theorem of Invertible Matrices implies that null$(A-3I) \neq \mathbf{0}$. That is $A\mathbf{x} = 3\mathbf{x}$ has a nontrivial solution, so 3 is an eigenvalue of A.

Since $A\mathbf{x} = 3\mathbf{x}$ implies $(A-3I)\mathbf{x} = \mathbf{0}$, we now compute null$(A-3I)$.

$$[A - 3I \mid \mathbf{0}] = \left[\begin{array}{cc|c} -1 & 2 & 0 \\ 2 & -4 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So, if $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue 3, then $x_1 = 2x_2$.

These eigenvectors are of the form $\mathbf{x} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix}$. That is nonzero multiples of $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Q: What does this tell us about null$(A-3I)$? What about E_3?

A: The above shows null$(A-3I) = \text{span}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = E_3$, the *eigenspace* of 3.

23. As in Example 4.5, we find all solutions λ of the equation $\det(A - \lambda I) = 0$.

$$\det(A - \lambda I) = \det \begin{bmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{bmatrix} = \lambda^2 - 5\lambda + 6$$

Since $\lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3) = 0$, the solutions are $\lambda = 2$ and $\lambda = 3$.

$\lambda = 2$: $A - 2I = \begin{bmatrix} 4-2 & -1 \\ 2 & 1-2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$

So, if $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue 2, then $x_2 = 2x_1$.

These eigenvectors are of the form $\mathbf{x} = \begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix}$. That is nónzero multiples of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

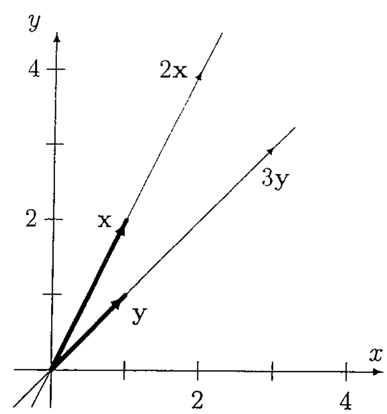
So, $E_2 = \text{null}(A - 2I) = \text{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$.

$\lambda = 3$: $A - 3I = \begin{bmatrix} 4-3 & -1 \\ 2 & 1-3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

So, if $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue 3, then $x_2 = x_1$.

These eigenvectors are of the form $\mathbf{x} = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$. That is nonzero multiples of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

So, $E_3 = \text{null}(A - 3I) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$.



3

25. As in Example 4.5, we find all solutions λ of the equation $\det(A - \lambda I) = 0$.

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 5 \\ 0 & 2-\lambda \end{bmatrix} = (2-\lambda)^2$$

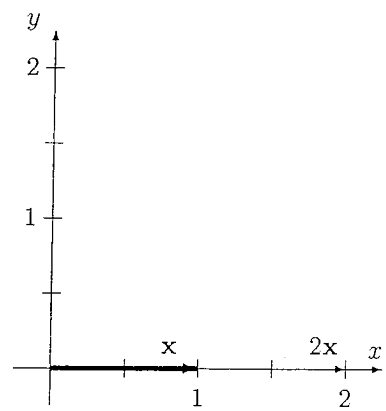
Since $(2-\lambda)^2 = (\lambda - 2)(\lambda - 2) = 0$, the solution is $\lambda = 2$.

$$\lambda = 2\colon\ A - 2I = \begin{bmatrix} 2-2 & 5 \\ 0 & 2-2 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

So, if $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector corresponding to 2, then $x_1 = t$, $x_2 = 0$.

These eigenvectors are of the form $\mathbf{x} = \begin{bmatrix} t \\ 0 \end{bmatrix}$. That is nonzero multiples of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

So, $E_2 = \operatorname{null}(A - 2I) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$.



27. As in Example 4.7, we find all solutions λ of the equation $\det(A - \lambda I) = 0$.

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{bmatrix} = \lambda^2 - 2\lambda + 2$$

Since $\lambda^2 - 2\lambda + 2 = 0$, the solutions are $\lambda = 1+i, 1-i$.

$$1+i\colon\ A - (1+i)I = \begin{bmatrix} 1-(1+i) & 1 \\ -1 & 1-(1+i) \end{bmatrix} = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$$

So, if $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector corresponding to $1+i$, then $x_1 = -ix_2 = -it$.

These eigenvectors are of the form $\mathbf{x} = \begin{bmatrix} -it \\ t \end{bmatrix}$. That is nonzero multiples of $\begin{bmatrix} -i \\ 1 \end{bmatrix}$.

So, $E_{1+i} = \operatorname{null}(A - (1+i)I) = \operatorname{span}\left(\begin{bmatrix} -i \\ 1 \end{bmatrix}\right)$.

$$1-i\colon\ A - (1-i)I = \begin{bmatrix} 1-(1-i) & 1 \\ -1 & 1-(1-i) \end{bmatrix} = \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$$

So, if $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector corresponding to $1-i$, then $x_1 = ix_2 = it$.

These eigenvectors are of the form $\mathbf{x} = \begin{bmatrix} it \\ t \end{bmatrix}$. That is nonzero multiples of $\begin{bmatrix} i \\ 1 \end{bmatrix}$.

So, $E_{1-i} = \operatorname{null}(A - (1-i)I) = \operatorname{span}\left(\begin{bmatrix} i \\ 1 \end{bmatrix}\right)$.

Dp43

(9)

3. (a) $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & -2-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (1-\lambda)(-2-\lambda)(3-\lambda).$

(b) $(1-\lambda)(-2-\lambda)(3-\lambda) = 0 \Leftrightarrow \lambda_1 = -2,\ \lambda_2 = 1,$ or $\lambda_3 = 3.$

(c) $A + 2I = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, so $E_{-2} = \operatorname{span}\left(\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}\right)$.

$A - I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, so $E_1 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$.

$A - 3I = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -5 & 1 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{10} \\ 0 & 1 & -\frac{1}{5} \\ 0 & 0 & 0 \end{bmatrix}$, so $E_3 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \\ 10 \end{bmatrix}\right)$.

(d) Each eigenvalue has algebraic and geometric multiplicity 1.

4. (a) $$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = -(1-\lambda)-(1-\lambda)-\lambda(1-\lambda)^2$$
$$= -2(\lambda-1)-\lambda(\lambda-1)^2 = (\lambda-1)(\lambda-2)(\lambda+1).$$

(b) $(\lambda-1)(\lambda-2)(\lambda+1)=0 \Leftrightarrow \lambda_1=-1,\ \lambda_2=1,\ \lambda_3=2.$

(c) $$A+I = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}, \text{ so } E_{-1} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}\right).$$
$$A-I = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \text{ so } E_{1} = \operatorname{span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right).$$
$$A-2I = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \text{ so } E_{2} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right).$$

(d) Each eigenvalue has algebraic and geometric multiplicity 1.

5. (a) $$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 2 & 0 \\ -1 & -1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = -(1-\lambda)+(1-\lambda)\left[(1-\lambda)(-1-\lambda)-(-2)\right]$$
$$= \lambda-1+\left(1-\lambda+\lambda^2-\lambda^3\right) = \lambda^2-\lambda^3 = \lambda^2(1-\lambda).$$

(b) $\lambda^2(1-\lambda)=0 \Leftrightarrow \lambda_1=\lambda_2=0,\ \lambda_3=1.$

(c) $$A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \text{ so } E_{0} = \operatorname{span}\left(\begin{bmatrix} 1 \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}\right).$$
$$A-I = \begin{bmatrix} 0 & 2 & 0 \\ -1 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ so } E_{1} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right).$$

(d) 0 has algebraic multiplicity 2 and geometric multiplicity 1,
while 1 has algebraic and geometric multiplicity 1.

14. Using induction and the proofs of 4.18(a) and (b), we will prove Theorem 4.18(c):
For any integer n if $A\mathbf{x} = \lambda\mathbf{x}$, then $A^n\mathbf{x} = \lambda^n\mathbf{x}$.

As suggested, we will also use the fourth *Remark* following Theorem 3.9 in Section 3.3:
If A is invertible and n is a positive integer, then $A^{-n} = (A^{-1})^n = (A^n)^{-1}$.

Since (a) gives us the result for positive integers, we proceed by induction on $-n$.

1: If $A\mathbf{x} = \lambda\mathbf{x}$, then $A^{-1}\mathbf{x} = \lambda^{-1}\mathbf{x}$.
Since $\lambda^{-1} = \frac{1}{\lambda}$, this is the statement of Theorem 4.18(b). So there is nothing to show.

k: If $A\mathbf{x} = \lambda\mathbf{x}$, then $A^{-k}\mathbf{x} = \lambda^{-k}\mathbf{x}$.
This is the induction hypothesis.

$k+1$: If $A\mathbf{x} = \lambda\mathbf{x}$, then $A^{-(k+1)}\mathbf{x} = \lambda^{-(k+1)}\mathbf{x}$.
This is the statement we must prove using the induction hypothesis.

$$A^{-(k+1)}\mathbf{x} \overset{\substack{\text{by}\\\text{Remark}}}{=} A^{-1}(A^{-k}\mathbf{x}) \overset{\substack{\text{by}\\\text{induc}}}{=} A^{-1}(\lambda^{-k}\mathbf{x}) = \lambda^{-k}(A^{-1}\mathbf{x}) \overset{\substack{\text{by}\\ n=1}}{=} \lambda^{-k}(\lambda^{-1}\mathbf{x}) = \lambda^{-(k+1)}\mathbf{x}$$

Q: Why does the *Remark* imply that $A^{-(k+1)} = A^{-1}A^{-k}$?
A: The remark implies both $A^{-k} = (A^k)^{-1}$ and $A^{-(k+1)} = (A^{k+1})^{-1}$.
So we need only show $A^{-1}A^{-k} = (A^{k+1})^{-1}$. That is, $(A^{-1}A^{-k})(A^{k+1}) = I$.
That is obvious since: $(A^{-1}A^{-k})(A^{k+1}) = A^{-1}(A^{-k}A^k)A = A^{-1}A = I$.

Q: What does the *Remark* and our work above suggest about integer exponents of A?
A: They behave precisely as we would hope. That is, like the exponents of real variables.

(optional)

20. Given $A^n = O$, we need to show if $A\mathbf{x} = \lambda\mathbf{x}$ then $\lambda = 0$. That is:
1) 0 is an eigenvalue of A and 2) if λ is an eigenvalue of A, then $\lambda = 0$.

To prove assertion 1 we make the following observation:

Q: What is the contrapositive of Theorem 4.16?
A: A is *not* invertible if and only if 0 *is* an eigenvalue of A.

Q: What does the contrapositive of Theorem 4.16 imply?
A: $\det A = 0$ if and only if 0 *is* an eigenvalue of A. Why? Because the contrapositive of Theorem 4.6 in Section 4.2 implies $\det A = 0$ if and only if A is *not* invertible.

So, to prove 0 is an eigenvalue of A it suffices to show $\det A = 0$.
That is, if $A^n = O$, then $\det A = 0$. So, 0 is an eigenvalue of A.

Since $(\det A)^n \overset{\substack{\text{Thm 4.8}\\\text{Sect 4.2}}}{=} \det(A^n) \overset{\substack{A^n=O\\\text{given}}}{=} \det O = 0$, $\det A = 0$. So, 0 is an eigenvalue of A.

Next we show if λ is an eigenvalue of A, then $\lambda = 0$.

If $A\mathbf{x} = \lambda\mathbf{x}$, then Theorem 4.18(c) implies $A^n\mathbf{x} = \lambda^n\mathbf{x} = O\mathbf{x} = \mathbf{0}$.
Since $\mathbf{x}$ is an eigenvector, $\mathbf{x} \neq \mathbf{0}$. So $\lambda^n\mathbf{x} = \mathbf{0}$ implies $\lambda^n = 0$. Therefore, $\lambda = 0$ as claimed.

(optional)

22. $A\mathbf{v} = \lambda\mathbf{v} \Rightarrow A\mathbf{v} - cI\mathbf{v} = \lambda\mathbf{v} - cI\mathbf{v} \Leftrightarrow (A - cI)\,\mathbf{v} = (\lambda - c)\,\mathbf{v}$.
So $\mathbf{v}$ is an eigenvector of $A - cI$ with corresponding eigenvalue $\lambda - c$.

(optional)

25. As noted in Theorem 4.17(d), $A \longrightarrow I$ if and only if A is invertible.

Q: If the conjecture of this exercise were true, what would that imply?
A: Since the only eigenvalue of I is 1, all invertible matrices would only have eigenvalue 1. This is clearly nonsense. However, they may be *related*.

Q: Let $\mathbf{x}$ be an eigenvector of A corresponding to eigenvalue λ.
If $A \xrightarrow{R_i \leftrightarrow R_j} B$, that is $B = E_{ij}A$, what goes wrong?
A: Since $B = E_{ij}A$, we have $B\mathbf{x} = E_{ij}(A\mathbf{x}) = \lambda(E_{ij}\mathbf{x})$.
So the components of $\mathbf{x}$ are interchanged and $\mathbf{x}$ fails to be an eigenvector for B.

Q: If $A \xrightarrow{kR_i} B$, what goes wrong?

Q: If $A \xrightarrow{R_i + kR_j} B$, what goes wrong?

Q: So, if $A \longrightarrow B$, we have seen their eigenvalues are not necessarily equal. However:
If $A \longrightarrow B$, is there a relationship among the eigenvalues and eigenvectors?
A: Hint: $2I$ has eigenvalue 2. Can this process be generalized? See Exercise 41.