

math 462/512. Test 1 solutions

1. Use the matrix exponent to solve the initial value problem

$$\dot{\mathbf{x}} = A\mathbf{x},$$

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -2 & 0 \end{pmatrix}, \quad \mathbf{x}(0) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \text{ Find the stable, unstable and center subspaces of}$$

$$\dot{\mathbf{x}} = A\mathbf{x}.$$

Hint: 2 and $1+i$ are the eigenvalues of A

$\lambda = 1+i$ is the eigenvalue $\Rightarrow \bar{\lambda} = 1-i$ is also the eigenvalue.

$$\lambda_1 = 2$$

$$(A - \lambda_1 I) \vec{v}_1 = \vec{0} \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda_2 = 1+i \Rightarrow \vec{w}_2 = \begin{pmatrix} 0 \\ -1/2 - i/2 \\ 1 \end{pmatrix} \Rightarrow \vec{u}_2 = \operatorname{Re} \vec{w}_2 = \begin{pmatrix} 0 \\ -1/2 \\ 1 \end{pmatrix}$$

$$\vec{v}_2 = \operatorname{Im} \vec{w}_2 = \begin{pmatrix} 0 \\ -1/2 \\ 0 \end{pmatrix}$$

$$P = (\vec{v}_1, \vec{v}_2, \vec{u}_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/2 & -1/2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\vec{x} = e^{At} \vec{x}_0 = P \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{t \cos t} - e^{t \sin t} & e^{t \sin t} \\ 0 & e^{t \sin t} & e^{t \cos t} \end{pmatrix} P^{-1} \vec{x}_0 = \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{t \cos t} + e^{t \sin t} & e^{t \sin t} \\ 0 & -2e^{t \sin t} & e^{t \cos t} - e^{t \sin t} \end{pmatrix} \vec{x}_0$$

$$E^s = \{\vec{0}\}$$

$$E^c = \{\vec{0}\}$$

$$E^u = \operatorname{Span}\{\vec{v}_1, \vec{v}_2, \vec{u}_2\}$$

2. Solve the initial value problem

$$\dot{\mathbf{x}} = A\mathbf{x},$$

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{x}(0) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \text{ Find the stable, unstable and center subspaces}$$

of $\dot{\mathbf{x}} = A\mathbf{x}$.

$$\text{Hint: } \det(A - \lambda I) = -(\lambda + 1)(\lambda - 1)^2.$$

$$\lambda = -1, \quad \lambda = 1 - \text{multiplicity } 2$$

$$\vec{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$(A - I)^2 \vec{v}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{v}_3 = 0 \Rightarrow \vec{v}_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S = P \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$N = A - S = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow N^2 = 0$$

$$SN = NS$$

$$\vec{x}(t) = e^{At} \vec{x}_0 = P \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{pmatrix} P^{-1} [I + Nt] \vec{x}_0$$

$$= \begin{pmatrix} e^t & 0 & e^{-t} \\ 0 & e^{-t} & e^{-t}e^t \\ 0 & 0 & e^t \end{pmatrix} \vec{x}_0$$

$$E^s = \text{span} \{ \vec{v}_1 \}$$

$$E^c = \{ \vec{0} \}$$

$$E^u = \text{span} \{ \vec{v}_2, \vec{v}_3 \}$$

3. Solve the nonhomogeneous linear system

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b},$$

$$\text{with } A = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 0 \\ t \end{pmatrix}.$$

$$\lambda = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$\vec{w} = \begin{pmatrix} -1/2 + i\frac{\sqrt{3}}{2} \\ 1 \end{pmatrix} \Rightarrow \vec{u} = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix} \quad \vec{v} = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ 0 \end{pmatrix}$$

$$P = (\vec{v}, \vec{u}) = \begin{pmatrix} \frac{\sqrt{3}}{2} & -1/2 \\ 0 & 1 \end{pmatrix} \quad P^{-1} = \begin{pmatrix} \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & 1 \end{pmatrix}$$

$$\Phi = P e^{t/2 \begin{pmatrix} \cos \frac{\sqrt{3}t}{2} & -\sin \frac{\sqrt{3}t}{2} \\ \sin \frac{\sqrt{3}t}{2} & \cos \frac{\sqrt{3}t}{2} \end{pmatrix}} P^{-1} = \frac{e^{-t/2}}{\sqrt{3}} \begin{pmatrix} \sqrt{3} \cos \frac{\sqrt{3}t}{2} - \sin \frac{\sqrt{3}t}{2} & -2 \sin \frac{\sqrt{3}t}{2} \\ 2 \sin \frac{\sqrt{3}t}{2} & \sqrt{3} \cos \frac{\sqrt{3}t}{2} + \sin \frac{\sqrt{3}t}{2} \end{pmatrix}$$

$$\Phi^{-1} = \Phi(-t) = \frac{e^{t/2}}{\sqrt{3}} \begin{pmatrix} \sqrt{3} \cos \frac{\sqrt{3}t}{2} + \sin \frac{\sqrt{3}t}{2} & 2 \sin \frac{\sqrt{3}t}{2} \\ -2 \sin \frac{\sqrt{3}t}{2} & \sqrt{3} \cos \frac{\sqrt{3}t}{2} - \sin \frac{\sqrt{3}t}{2} \end{pmatrix}$$

$$\vec{x}(t) = \Phi(t) \underbrace{\Phi^{-1}(0)}_{I} \vec{x}_0 + \int_0^t \Phi(t) \Phi^{-1}(\tau) \vec{b}(\tau) d\tau$$

$$= \Phi(t) \vec{x}_0 + \Phi(t) \int_0^t e^{-\tau/2} \begin{pmatrix} 2 \sin \frac{\sqrt{3}\tau}{2} \\ \sqrt{3} \cos \frac{\sqrt{3}\tau}{2} + \sin \frac{\sqrt{3}\tau}{2} \end{pmatrix} e^{\tau/2} d\tau$$

$$= \Phi(t) \vec{x}_0 + \Phi(t) \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \Phi(t) e^{-t/2} \begin{pmatrix} -(t+1) \cos \frac{\sqrt{3}t}{2} - \frac{1}{\sqrt{3}}(t+1) \sin \frac{\sqrt{3}t}{2} \\ \cos \frac{\sqrt{3}t}{2} + \frac{1}{\sqrt{3}}(2+t) \sin \frac{\sqrt{3}t}{2} \end{pmatrix}$$

4. Determine all solution(s) of the initial value problem

$$\dot{x} = |x|^{2/3}, \quad x(0) = 0.$$

Trivial sol. $x(t) = 0$

$$\underline{x > 0}$$

$$\Rightarrow \dot{x} = x^{2/3}$$

$$\frac{dx}{x^{2/3}} = dt$$

$$3(x^{1/3} - \underbrace{x_0}_{0}^{1/3}) = t$$

$$\Rightarrow x = \frac{t^3}{27} > 0 \Rightarrow t > 0$$

$$\underline{x < 0} \quad \dot{x} = (-x)^{2/3}, \quad y = -x$$

$$\Rightarrow \dot{y} = -y^{2/3}$$

$$\frac{dy}{y^{2/3}} = -dt$$

$$3(y^{1/3} - \underbrace{y_0}_{0}^{1/3}) = -t$$

$$y = -\frac{t^3}{27} \Rightarrow$$

$$x = \frac{t^3}{27} < 0 \Rightarrow t < 0$$

All together $\Rightarrow$ possible solutions:

$$(1) \quad x = \begin{cases} t^3/27, & t > 0 \\ 0, & t \leq 0 \end{cases}$$

$$(2) \quad x = \begin{cases} 0, & t > 0 \\ t^3/27, & t \leq 0 \end{cases}$$

$$(3) \quad x = 0 \quad \forall t$$

$$(4) \quad x = \frac{t^3}{27} \quad \forall t$$

5. Use the mathematical induction to find the solution of the initial value problem

$$\dot{x} = -tx, \quad x(0) = 3$$

by the (Picard's) method of successive iterations.

Pf by induction

$$(1) \quad u_0 = x(0) = 3$$

$$u_1 = 3 + \int_0^t 3s \, ds = 3 + \frac{3}{2} t^2$$

$$(2) \quad \text{Assume } u_n = 3 \left(1 - \frac{t^2}{2} + \frac{t^4}{2 \cdot 2!} - \dots + (-1)^n \frac{t^{2n}}{2^n n!} \right)$$

$$= 3 + \int_0^t 3(-s) \left(1 - \frac{s^2}{2} + \frac{(-1)^n s^{2n}}{2^n n!} \right) ds$$

$$= 3 + 3 \left(-\frac{s^2}{2} + \frac{s^4}{2 \cdot 4} - \dots + (-1)^n \frac{(-1) s^{2n+2}}{2^n n! (2n+2)} \right) \Big|_0^t$$

$$= 3 \left(1 - \frac{t^2}{2} + \dots + \frac{(-1)^{n+1} t^{2(n+1)}}{2^{n+1} (n+1)!} \right)$$

$$\Rightarrow \lim_{n \rightarrow \infty} u_n = 3 e^{-\frac{t^2}{2}}$$

$$x = x(0) e^{-\frac{t^2}{2}} = 3 e^{-\frac{t^2}{2}}$$

check by explicit solution: