math 462/512 HW01 Solutions

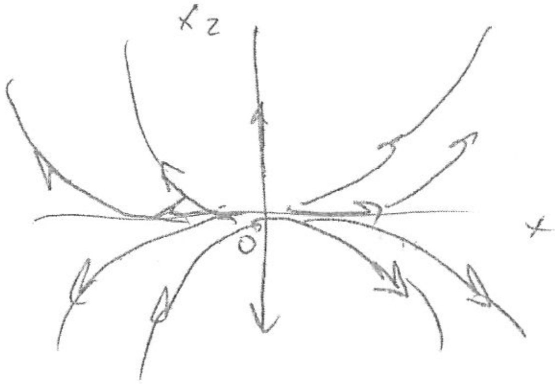
1.1 (1) (b)

$$\begin{cases} \dot{x}_1 = x_1 \\ \dot{x}_2 = 2x_2 \end{cases} \Rightarrow \begin{matrix} x_1 = x_{1,0} e^{t} \\ x_2 = x_{2,0} e^{2t} \end{matrix}$$

$$x_1^2 = x_{1,0}^2 e^{2t} = \frac{x_{1,0}^2 x_2}{x_{2,0}}$$

$\Rightarrow$ Solution curves lie on the parabolas ~~y~~ $x_2 = \frac{x_{2,0}}{x_{1,0}^2} x_1^2$

(except ~~those~~ those lying on x_1, x_2 axes.



1.1 (1d)

$$\begin{cases} \dot{x}_1 = -x_2 \\ \dot{x}_2 = x_1 \end{cases} \qquad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

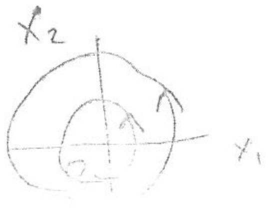
$$\lambda = \pm i$$

$\lambda = i$

$\vec{v} = \begin{pmatrix} i \\ 1 \end{pmatrix}$

$$\vec{x} = c_1 \operatorname{Re}\left(e^{it}\begin{pmatrix} i \\ 1 \end{pmatrix}\right) + c_2 \operatorname{Im}\left(e^{it}\begin{pmatrix} i \\ 1 \end{pmatrix}\right)$$

$$= c_1 \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$



1.1.6

(a)

$$\dot{\vec{w}}(t) = a\dot{\vec{u}}(t) + b\dot{\vec{v}}(t)$$

$$= a A\vec{u}(t) + b A\vec{v}(t) = A(a\vec{u}(t) + b\vec{v}(t))$$

$$= A\vec{w}(t) \text{ for } \forall t \in \mathbb{R} \quad \boxed{} \text{ i.e. sol.}$$

(b) $\vec{u}(t) = \begin{pmatrix} e^t \\ 0 \end{pmatrix}$

$\vec{v}(t) = \begin{pmatrix} 0 \\ e^{-2t} \end{pmatrix}$

General sol $\dot{\vec{x}} = A\vec{x}$ is

given by $\vec{x}(t) = x_0\vec{u}(t) + y_0\vec{v}(t)$

$= \begin{pmatrix} x_0 e^t \\ y_0 e^{-2t} \end{pmatrix}$, where $\vec{x}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

1.2.1a

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

$$\lambda_1 = 2, \quad \lambda_2 = 4$$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

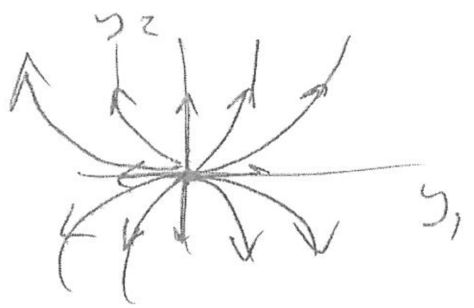
$$P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \Rightarrow P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

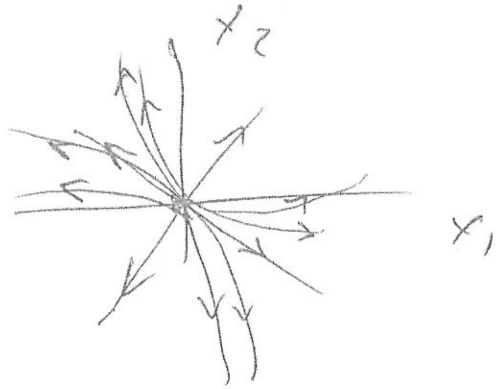
$$B = P^{-1}AP = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

$$\vec{y}(t) = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{4t} \end{pmatrix} \vec{y}_0$$

$$\vec{x}(t) = P\begin{pmatrix} e^{2t} & 0 \\ 0 & e^{4t} \end{pmatrix} P^{-1}\vec{x}_0$$

$$= \frac{1}{2}\begin{pmatrix} e^{2t}+e^{4t} & e^{4t}-e^{2t} \\ e^{4t}-e^{2t} & e^{4t}+e^{2t} \end{pmatrix} \vec{x}_0$$





1.2 2 $\quad \dot{\vec{x}} = A\vec{x}$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$\lambda_1 = 1$, $\vec{v}_1 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ | $\lambda_2 = 2$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ | $\lambda_3 = -1$, $\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\vec{y} = \begin{pmatrix} e^{t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix} \vec{y}_0 \qquad \vec{x}(t) = \frac{1}{2} \begin{pmatrix} 2e^{2t} & 0 & 0 \\ 2(e^{2t} - e^{t}) & 2e^{2t} & 0 \\ e^{t} - e^{-t} & 0 & 2e^{-t} \end{pmatrix} \vec{x}_0$$

$E^s = \mathrm{Span}\{\vec{v}_3\}$.

$E^u = \mathrm{Span}\{\vec{v}_1, \vec{v}_2\}$.

1.2.3c $\quad \dddot{x} - 2\ddot{x} - \dot{x} + 2x = 0$

$x_1 = x$

$x_2 = \dot{x}$

$x_3 = \ddot{x}$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{pmatrix}$$

$\dot{x}_1 = x_2$

$\dot{x}_2 = x_3$

$\dot{x}_3 = -2x_1 + x_2 + 2x_3$

$\lambda = 2 \qquad \lambda = -1 \qquad \lambda = 1$

$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow P = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 4 & 1 & 1 \end{pmatrix}$

$E = \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{t} \end{pmatrix}, \quad \vec{x} = P E P^{-1} \vec{x}_0 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x \\ \dot{x} \\ \ddot{x} \end{pmatrix} =$

$$= \begin{pmatrix} \frac{e^{-t}}{3} + e^{t} - \frac{e^{2t}}{3} & -\frac{1}{2} e^{-t} + \frac{e^{t}}{2} & \frac{e^{-t}}{6} - \frac{e^{t}}{2} + \frac{e^{2t}}{3} \\ -\frac{e^{-t}}{3} + e^{t} - \frac{2e^{2t}}{3} & \frac{e^{-t}}{2} + \frac{e^{t}}{2} & -\frac{e^{-t}}{6} - \frac{e^{t}}{2} + \frac{2}{3} e^{2t} \\ e^{t} + \frac{e^{-t}}{3} - \frac{4e^{2t}}{3} & -\frac{e^{-t}}{2} + \frac{e^{t}}{2} & \frac{e^{-t}}{6} - \frac{e^{t}}{2} + \frac{4e^{2t}}{3} \end{pmatrix} \vec{x}_0$$

Alternatively we can write general sol as linear combination of particular solutions:

$$\vec{x} = \begin{pmatrix} x \\ \dot{x} \\ \ddot{x} \end{pmatrix} = c_1 e^{2t} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + c_3 e^{t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

1.2.6 $\quad A \in \mathbb{R}^{n \times n}, \quad \varphi(t, \vec{x}_0) = P \begin{pmatrix} e^{\lambda_1 t} & 0 \cdots & 0 \\ 0 & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix} P^{-1} \vec{x}_0$

$\lim_{\vec{y}_0 \to \vec{x}_0} \varphi(t, \vec{y}) = \varphi(t, \vec{x}_0)$ since $\lim_{\vec{y}_0 \to \vec{x}_0} \vec{y}_0 = \vec{x}_0$

1.4.4 $\quad A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{pmatrix}$

From 1.2.2 : $\quad e^{At} \vec{x}_0 = P \operatorname{diag}(e^{\lambda_i t}) P^{-1} \vec{x}_0$

$$= \frac{1}{2} \begin{pmatrix} 2e^{2t} & 0 & 0 \\ 2(e^{2t} - e^{t}) & 2e^{2t} & 0 \\ e^{t} - e^{-t} & 0 & 2e^{-t} \end{pmatrix} \vec{x}_0$$

where

$P = \begin{pmatrix} 2 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$

$P^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$

1.7.1.a

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$$

$$\lambda_1 = \lambda_2 = 1$$

$$\Rightarrow \quad S = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$$

and $N = A - S = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$, S and N commute.

$$N^2 = 0$$

$$\vec{x}(t) = P \begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix} P^{-1} \left[I + Nt \right] \vec{x}_0$$

$$= e^t \begin{pmatrix} 1-t & 1 \\ -t & 1+t \end{pmatrix} \vec{x}_0$$

1.7.2.b $A = \begin{pmatrix} -1 & 1 & -2 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$

$\lambda_1 = \lambda_2 = -1$

$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\lambda_3 = 1$

$\vec{v}_3 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

Second generalized eigenvector:

$$(A + I)^2 \vec{v}_2 = \vec{0}$$

$$\Rightarrow (A+I)^2 \vec{v}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 8 \\ 0 & 0 & 4 \end{pmatrix} \vec{v}_2 = \vec{0}$$

$$\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$ (linearly independent from $\vec{v}_1$)

$$\Rightarrow S = P \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix} P^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{pmatrix},$$

where $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$

$$\Rightarrow N = A - S = \begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$SN = NS$, $N^2 = 0 \Rightarrow$

$$\vec{x}(t) = e^{At}\vec{x}_0 = e^{St}[I + Nt]$$

$$= P \operatorname{diag}\begin{pmatrix} e^{-t} & & \\ & e^{-t} & \\ & & e^{t} \end{pmatrix} P^{-1} [I + Nt] \vec{x}_0$$

$$= \begin{pmatrix} e^{-t} & te^{-t} & -2te^{-t} \\ 0 & e^{-t} & 2(e^{t} - e^{-t}) \\ 0 & 0 & e^{t} \end{pmatrix} \vec{x}_0$$

1.5.1a $\delta = \det A$, $\tau = \text{Trace } A$

$\delta = -2 < 0 \Rightarrow$ saddle at the origin

1.5.1b $\delta = 8$, $\tau = 6$, $\tau^2 - 4\delta = 4 > 0$

$\Rightarrow$ unstable node at the origin

1.5.1c $\delta = 2$, $\tau = 0 \Rightarrow$ center at the origin.

1.5.5 $\ddot{x} + a\dot{x} + bx = 0$

$x = x_1$

$\dot{x} = x_2$

$\dot{x}_1 = x_2$

$\dot{x}_2 = -a x_2 - b x_1$

$$A = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}$$

$\tau^2 - 4\delta = a^2 - 4b$

$\left.\begin{aligned} \delta &= b \\ \tau &= -a \end{aligned}\right\} \Rightarrow$ (1) if $b < 0$ $\Rightarrow$ saddle

(2) $b > 0$ and $a^2 - 4b \geq 0$, $a < 0$ – unstable node (source)

(3) $b > 0$, $a^2 - 4b \geq 0$, $a > 0$ – stable node (Sink)

(4) $b > 0$ and $a^2 - 4b < 0$ and $a \neq 0 \Rightarrow$ focus (~~center~~)

stable for $a > 0$ (spiral sink)

and unstable for $a < 0$ (spiral source)

(5) $b > 0,\ a = 0 \Rightarrow$ center

(6) $a = 0 = b$ - degenerate critical point

1.3.1 $A = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$

(a) $\|A\| = \max\limits_{|\vec{x}| \le 1} |A\vec{x}| = \max\limits_{|\vec{x}| \le 1} \sqrt{4x^2 + 9y^2} \le$

$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

$|\vec{x}|^2 = x^2 + y^2 \le 1$

$\Rightarrow y^2 \le 1 - x^2$

$\le \max\limits_{|\vec{x}| \le 1} \sqrt{4x^2 + 9(1 - x^2)}$

$= \max\limits_{|\vec{x}| \le 1} \sqrt{9 - 5x^2} = \sqrt{9} = 3$

max achieved for $x = 0,\ y \ne 0$

(b) $A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$ use hint for (c).

$|A\vec{x}|^2 = \left| \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right| = \left((x+2y)^2 + y^2 \right)$

$= x^2 + 4xy + 5y^2$ - need to maximize subject to constraint $x^2 + y^2 = 1$

$\Rightarrow y = \pm\sqrt{1 - x^2}$

$\Rightarrow f := x^2 + 4xy + 5y^2 = x^2 + 4x\left(\pm\sqrt{1-x^2}\right) + 5 - 5x^2$

$= 5 - 4x^2 + 4x\left(\pm\sqrt{1-x^2}\right)$

Maximum at:

$$f' = -8x = 4\left(\pm\sqrt{1-x^2}\right) \mp \frac{4x^2}{\sqrt{1-x^2}} = 0$$

$$\Rightarrow \quad -8x\sqrt{1-x^2} \pm 4(1-x^2) \mp 4x^2 = 0$$

$$-8x\sqrt{1-x^2} \pm (4-8x^2) = 0$$

$$2x\sqrt{1-x^2} = \pm(1-2x^2)$$

$$4x^2(1-x^2) = 1-4x^2+4x^4$$

$$8x^4 - 8x^2 + 1 = 0$$

choose +

$$x^2 = \frac{2\pm\sqrt{2}}{4} \Rightarrow y^2 = 1-x^2 = \frac{+2\mp\sqrt{2}}{4}$$

$$\Rightarrow \|A\|^2 = \max\left(x^2+4xy+5y^2\right)\Big|_{x^2=\frac{2\pm\sqrt{2}}{4},\ y^2=\frac{2\mp\sqrt{2}}{4}}$$

$$= \max\left(\frac{2\pm\sqrt{2}}{4} + 4\sqrt{\frac{2\pm\sqrt{2}}{4}}\sqrt{\frac{2\mp\sqrt{2}}{4}} + 5\,\frac{2\mp\sqrt{2}}{4}\right)$$

$$= \max\left(\frac{2\pm\sqrt{2}}{4} + \sqrt{4-2} + 5\,\frac{2\mp\sqrt{2}}{4}\right)$$

$$= \max\left(\frac{2\pm\sqrt{2}+4\sqrt{2}+10\mp5\sqrt{2}}{4}\right) = \frac{12+\sqrt{2}(\mp 4+4)}{4}$$

$$= 3+2\sqrt{2} \Rightarrow \boxed{\|A\| = \sqrt{3+2\sqrt{2}} \approx 2.41421\ldots}$$

Alternative method:

$$AA^T = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$$

$$\det(AA^T - \lambda I) = 0 \Rightarrow \lambda = 3 \pm 2\sqrt{2}$$

$$\|A\| = \sqrt{\max \lambda} = \sqrt{3 + 2\sqrt{2}}$$

(c) $A = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$

$$|A\vec{x}|^2 = 26x^2 + 10xy + y^2 \quad \text{with } x^2 + y^2 = 1, \ |x| \le 1$$

$$= 26x^2 \pm 10x\sqrt{1-x^2} + 1 - x^2$$

$$= 25x^2 \pm 10x\sqrt{1-x^2} + 1 = f(x)$$

to find max:

$$\frac{df}{dx} = 50x \pm 10\sqrt{1-x^2} \mp \frac{10x^2}{\sqrt{1-x^2}} = 0$$

$$\Rightarrow 5x = \mp\left(\frac{1-x^2-x^2}{\sqrt{1-x^2}}\right)$$

$$\Rightarrow 25x^2(1-x^2) = (1-2x^2)^2$$

$$\Rightarrow \quad x^2 = \frac{1}{2} \pm 5\frac{\sqrt{29}}{58}$$

$$\Rightarrow y^2 = \frac{1}{2} \mp 5\frac{\sqrt{29}}{58}$$

$$\|A\| = \sqrt{\frac{27}{2} + \frac{5\sqrt{29}}{2}} \approx 5.1925\ldots$$

Alternative method:

$$AA^T = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$$

$$\det\left(AA^T - \lambda I\right) = 0 \Rightarrow \lambda = 3 \pm 2\sqrt{2}$$

$$\|A\| = \sqrt{\max \lambda} = \sqrt{3 + 2\sqrt{2}}$$

(c) Similar to (b)!

$$|A\vec{x}|^2 = 26x^2 + 10xy + y^2 \text{ with } x^2 + y^2 = 1$$

$$\Rightarrow \|A\| = \max\sqrt{\frac{27 \pm \sqrt{725}}{2}} = \sqrt{\frac{27 + \sqrt{725}}{2}} \simeq 5.19\ldots$$

Alternative:

$$\det(AA^T - \lambda I) = 0 \Rightarrow \lambda = \frac{27 \pm \sqrt{725}}{2} = \frac{27 \pm 5\sqrt{29}}{2}$$

$$\Rightarrow \|A\| = \sqrt{\max \lambda} = \sqrt{\frac{27 + \sqrt{725}}{2}}.$$

$$\simeq 5.1925$$

1.3.3 If T is invertibl

$\Rightarrow \exists\ T^{-1}$ such that $TT^{-1} = I$.

$\Rightarrow \|TT^{-1}\| = 1$.

By lemma of Sec 1.3. :

$$1 = \|TT^{-1}\| \leq \|T\|\,\|T^{-1}\|$$

$$\Rightarrow \|T\| > 0,\ \|T^{-1}\| > 0$$

and $\boxed{\|T^{-1}\| \geq \frac{1}{\|T\|}}$ $\square$

1.3.7(a) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$$e^A = \begin{pmatrix} e^1 & 0 & 0 \\ 0 & e^2 & 0 \\ 0 & 0 & e^3 \end{pmatrix}.$$