

1.6.2

$$A = \begin{pmatrix} 0 & -2 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\lambda = 1 \pm i$$

$$\vec{w} = \begin{pmatrix} 1-i \\ -1 \\ 0 \end{pmatrix}$$

$$\lambda_3 = -2$$

$$\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$P = (\vec{w}, \vec{w}, \vec{v}_3) = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \vec{x}(t) = P \begin{pmatrix} e^{t \cos t} & -e^t \sin t & 0 \\ e^{t \sin t} & e^t \cos t & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix} P^{-1} \vec{x}_0$$

$$= \begin{pmatrix} e^t (\cos t - \sin t) & -2e^t \sin t & 0 \\ e^t \sin t & e^t (\sin t + \cos t) & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix} \vec{x}_0$$

1.6.4

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$$A = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\lambda_1 = -1 + i$$

$$\lambda_3 = (1 + i)$$

$$\vec{w}_1 = \text{vec} \begin{pmatrix} 1 \\ -i \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{w}_3 = \begin{pmatrix} 0 \\ 0 \\ 1-i \\ -1 \end{pmatrix}$$

$$P = (\text{Im } w_1, \text{Re } w_1, \text{Im } w_3, \text{Re } w_3)$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad P^{-1} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\vec{x}(t) = P \begin{pmatrix} e^{-t} R & 0 \\ 0 & e^t R \end{pmatrix} P^{-1} \vec{x}_0$$

$$R = \begin{pmatrix} e^{-t} \cos t & -e^{-t} \sin t & 0 & 0 \\ e^{-t} \sin t & e^{-t} \cos t & 0 & 0 \\ 0 & 0 & e^t (\cos t - \sin t) & -2e^t \sin t \\ 0 & 0 & e^t \sin t & e^t (\sin t + \cos t) \end{pmatrix} \vec{x}_0$$

$$\text{where } R = \begin{pmatrix} \cos t - \sin t & \\ \sin t & \cos t \end{pmatrix}$$

p. 37

1.7.3

$$(d) \lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 1+i, \lambda_4 = 1-i$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \text{are} & & & \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\lambda_1 = 1, \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda_2 = -1, \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda_3 = 1+i, \vec{w}_3 = \begin{pmatrix} 0 \\ 0 \\ i \\ 1 \end{pmatrix} \Rightarrow \vec{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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$$\vec{x}(t) = e^{At} \vec{x}_0$$

$$= P \begin{pmatrix} e^t & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 \\ 0 & 0 & e^{t \cos t} & -e^{t \sin t} \\ 0 & 0 & e^{t \sin t} & e^{t \cos t} \end{pmatrix} P^{-1} \vec{x}_0$$

$$= \begin{pmatrix} \cosh t & \sinh t & 0 & 0 \\ \sinh t & \cosh t & 0 & 0 \\ 0 & 0 & e^{t \cos t} & -e^{t \sin t} \\ 0 & 0 & e^{t \sin t} & e^{t \cos t} \end{pmatrix} \vec{x}_0$$

$$\textcircled{f} A = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\lambda_1 = \lambda_2 = 1 + i \quad \text{multiplicity 2}$$

$$\vec{w}_1 = \begin{pmatrix} i \\ 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \vec{v}_1 = \text{Im}(\vec{w}_1) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{v}_2 = \text{Re}(\vec{w}_1) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$(A - (1+i)I)^2 \vec{w}_2 = 2 \begin{pmatrix} -1 & i & -i & -1 \\ -i & -1 & 1 & -i \\ 0 & 0 & -1 & i \\ 0 & 0 & 0 & -i-1 \end{pmatrix} \vec{w}_2 = \vec{0}$$

$$\Rightarrow \vec{w}_2 = \begin{pmatrix} i \\ 1 \\ i \\ 1 \end{pmatrix} \Rightarrow \vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{v}_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

(5)

$$\Rightarrow P = (\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$S = P \operatorname{diag} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} P^{-1} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\Rightarrow N = A - S = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$SN = NS, \quad N^2 = 0$$

$$\Rightarrow \vec{x}(t) = e^{At} \vec{x}_0 = e^t P \operatorname{diag} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} P^{-1} \\ \times [I + Nt] \vec{x}_0 = e^t \begin{pmatrix} \cos t & -\sin t & t \cos t & -t \sin t \\ \sin t & \cos t & t \sin t & t \cos t \\ 0 & 0 & \cos t & -\sin t \\ 0 & 0 & \sin t & \cos t \end{pmatrix} \times \vec{x}_0.$$

⑥

1.8.2

①, ②, ③, ④ - already in Jordan Canonical form.

1.8.9 ①

$$\begin{bmatrix} a_1 & -b_1 & 0 & 0 \\ b_1 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & -b_2 \\ 0 & 0 & b_2 & a_2 \end{bmatrix}, \begin{bmatrix} a & -b & 1 & 0 \\ b & a & 0 & 1 \\ 0 & 0 & a & -b \\ 0 & 0 & b & a \end{bmatrix}, \begin{bmatrix} a & -b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix}, \begin{bmatrix} a & -b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}.$$

⑥ $x(t) = P \begin{bmatrix} e^{a_1 t} R_{b_1 t} & 0 \\ 0 & e^{a_2 t} R_{b_2 t} \end{bmatrix} P^{-1} x_0, x(t) = e^{at} P \begin{bmatrix} R_{bt} & t R_{bt} \\ 0 & R_{bt} \end{bmatrix} P^{-1} x_0.$

$$x(t) = P \begin{bmatrix} e^{at} R_{bt} & 0 \\ 0 & e^{\lambda_1 t} & 0 \\ 0 & 0 & e^{\lambda_2 t} \end{bmatrix} P^{-1} x_0, x(t) = P \begin{bmatrix} e^{at} R_{bt} & 0 \\ 0 & e^{\lambda_1 t} & t e^{\lambda_1 t} \\ 0 & 0 & e^{\lambda_2 t} \end{bmatrix} P^{-1} x_0.$$

$$R_{bt} = \begin{pmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{pmatrix}$$

1.8.7

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$$B = \begin{pmatrix} \lambda & 1 & & 0 \\ 0 & \lambda & 1 & \\ & & \ddots & \ddots \\ & & & \lambda \end{pmatrix} \in \mathbb{R}^{m \times m}$$

$$Q = \text{diag}(1, \varepsilon, \dots, \varepsilon^{m-1})$$

$$B = \lambda I + N, \quad N = \begin{pmatrix} 0 & 1 & 0 & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$$

$$Q^{-1} = \begin{pmatrix} 1 & & & 0 \\ & \frac{1}{\varepsilon} & & \\ & & \frac{1}{\varepsilon^2} & \\ & 0 & & \ddots & \\ & & & & \frac{1}{\varepsilon^{m-1}} \end{pmatrix}$$

$$\Rightarrow Q^{-1} B Q = Q^{-1} \lambda I Q + Q^{-1} N Q$$

$$= \lambda I + Q^{-1} N Q$$

here

$$Q^{-1} N Q = \begin{pmatrix} 0 & 1 & 0 & & \\ 0 & 0 & \frac{1}{\varepsilon} & 0 & \\ 0 & & & \ddots & \\ 0 & & & & \frac{1}{\varepsilon^{m-2}} \\ 0 & & & & 0 \end{pmatrix} Q$$

because N is
a shift matrix

$$= \begin{pmatrix} 0 & \varepsilon & 0 & & \\ 0 & 0 & \varepsilon & & \\ \vdots & & & \ddots & \\ 0 & & & & \varepsilon \\ 0 & & & & 0 \end{pmatrix} = \varepsilon N$$

1.8.10

If the elementary blocks in the Jordan form of A have the form $B = \text{diag} [\lambda, \dots, \lambda]$ or $B = \text{diag} [D, \dots, D]$ where D is a 2×2 matrix of the form in the theorem stated in this section, then each coordinate in the solution $x(t)$ of the initial value problem (4) will be a linear combination of functions of the form $e^{\lambda t}$, $e^{at} \cos bt$ or $e^{at} \sin bt$. Furthermore, if all of the eigenvalues of A have non-positive real part, i.e., if $\lambda \leq 0$ and $a \leq 0$ in the above forms, then each of the coordinates of $x(t)$ are bounded by constants (depending on $x_0 \in \mathbb{R}^n$) for all $t \geq 0$ and therefore for each $x_0 \in \mathbb{R}^n$, there exists a positive constant M such that $|x(t)| \leq M$ for all $t \geq 0$.

1.9.2 (8) $A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$$\lambda_1 = -1$$

$$\vec{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_{2,3} = \pm i$$

$$\vec{w}_2 = \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$E^s = \text{span}\{\vec{v}_1\},$$

$$E^u = \{0\}$$

$$E^c = \text{span}\{\vec{v}_2, \vec{v}_3\}.$$

(1.9.2) d

$$\lambda_1 = 2, \quad \lambda_2 = -1 \text{ (multiplicity 2)}$$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\vec{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$E^s = \text{span} \{ \vec{v}_2, \vec{v}_3 \}$$

$$E^u = \text{span} \{ \vec{v}_1 \}$$

$$E^c = \{ \vec{0} \}.$$

(1.3.2)

By definition, $\|T\| = \max_{|x| \leq 1} |T(x)|$. Thus, $\|T\| \geq \max_{|x|=1} |T(x)|$. But $\max_{|x|=1} |T(x)| = \sup_{x \neq 0} \frac{|T(x)|}{|x|}$

since if $|x| = a$ and we set $y = x/a$ for $x \neq 0$, then $|y| = |x|/a = 1$ and since T is linear,

$$\sup_{x \neq 0} \frac{|T(x)|}{|x|} = \sup_{x \neq 0} \frac{|T(x)|}{a} = \sup_{x \neq 0} \left| T\left(\frac{x}{a}\right) \right| = \max_{|y|=1} |T(y)|. \text{ Thus, } \|T\| \leq \sup_{0 < |x| \leq 1} \frac{|T(x)|}{|x|} \leq \sup_{x \neq 0} \frac{|T(x)|}{|x|} = \max_{|x|=1} |T(x)|.$$

It follows that $\|T\| = \max_{|x|=1} |T(x)| = \sup_{x \neq 0} |T(x)|/|x|$.

1.9.3

$$\frac{d}{dt} \vec{x} = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 2 & 0 & 6 \end{pmatrix} \vec{x}$$

①

$$\lambda_1 = 2i \quad \bar{\lambda}_1 = -2i \quad \vec{w}_1 = u_1 + i v_1 = \begin{pmatrix} 10 \\ 0 \\ -3 \end{pmatrix} + i \begin{pmatrix} 0 \\ 10 \\ -1 \end{pmatrix}$$

$$\lambda_3 = 6 \quad \vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 0 & 10 & 0 \\ 10 & 0 & 0 \\ -1 & -3 & 1 \end{pmatrix} = (\vec{w}_1 \quad \bar{w}_1 \quad \vec{v}_3)$$

$$P^{-1} = \frac{1}{10} \begin{pmatrix} 0 & 10 & 0 \\ 1 & 0 & 0 \\ 3 & 1 & 10 \end{pmatrix}$$

$$S = P^{-1} A P = \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

$$D = e^{S t} = \begin{pmatrix} \cos 2t & -\sin 2t & 0 \\ \sin 2t & \cos 2t & 0 \\ 0 & 0 & e^{6t} \end{pmatrix}$$

$$\vec{x} = P D P^{-1} = \begin{pmatrix} \cos 2t & \sin 2t & 0 \\ -\sin 2t & \cos 2t & 0 \\ \frac{1}{10} (3e^{6t} - 3\cos 2t + \sin 2t) & \frac{e^{6t}}{10} - \frac{\cos 2t}{10} - \frac{3\sin 2t}{10} & e^{6t} \end{pmatrix} \vec{x}_0$$

②

①

$$E^s = \{\vec{0}\}$$

$$E^u = \text{Span}\{\vec{v}_3\} = \text{Span}\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$$

$$E^c = \text{Span}\{\vec{v}_1, \vec{u}_1\} = \text{Span}\left\{\begin{pmatrix} 0 \\ 10 \\ -1 \end{pmatrix}, \begin{pmatrix} 10 \\ 0 \\ -3 \end{pmatrix}\right\}$$

$$n=t$$

$$e^{At} \vec{c}_3 \vec{v}_3 = P D P^{-1} \vec{v}_3 = \begin{pmatrix} \cos 2t & + \sin 2t \\ -\sin 2t & \cos 2t \\ \dots & \dots \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{6t}$$

$$= \vec{c}_3 \begin{pmatrix} 0 \\ 0 \\ e^{6t} \end{pmatrix} \in E^u$$

$$e^{At} (c_1 \vec{v}_1 + c_2 \vec{u}_1)$$

$$= \begin{pmatrix} \cos 2t & \sin 2t & 0 \\ -\sin 2t & \cos 2t & 0 \\ \frac{1}{10}(3e^{6t} - 3\cos 2t + \sin 2t) & \frac{e^{6t}}{10} - \frac{\cos 2t}{10} - \frac{3\sin 2t}{10} & e^{6t} \end{pmatrix} \left(c_1 \begin{pmatrix} 10 \\ 0 \\ -3 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 10 \\ -1 \end{pmatrix} \right)$$

$$= c_1 \begin{pmatrix} 10\sin 2t \\ 10\cos 2t \\ \frac{e^{6t}}{10} - \frac{\cos 2t}{10} - \frac{3\sin 2t}{10} - e^{6t} \end{pmatrix} + c_2 \begin{pmatrix} 10\cos 2t \\ -10\sin 2t \\ (3e^{6t} - 3\cos 2t + \sin 2t) - 3e^{6t} \end{pmatrix}$$

$$= \sin(2t) \begin{pmatrix} 10c_1 \\ -10c_2 \\ -c_1 - 3c_2 \end{pmatrix} + \cos(2t) \begin{pmatrix} 10c_2 \\ 10c_1 \\ -c_1 - 3c_2 \end{pmatrix} \quad (3)$$

$$= \sin(2t) c_1 \begin{pmatrix} 10 \\ 0 \\ -3 \end{pmatrix} + \sin(2t) c_2 \begin{pmatrix} 0 \\ -10 \\ 1 \end{pmatrix} \\ + \cos(2t) c_1 \begin{pmatrix} 0 \\ 10 \\ -1 \end{pmatrix} + \cos(2t) c_2 \begin{pmatrix} 10 \\ 0 \\ -3 \end{pmatrix}$$

$\begin{pmatrix} 10 \\ 0 \\ -3 \end{pmatrix} = \vec{u}_1$
 $\begin{pmatrix} 0 \\ -10 \\ 1 \end{pmatrix} = -\vec{v}_1$
 $\begin{pmatrix} 0 \\ 10 \\ -1 \end{pmatrix} = \vec{v}_1$
 $\begin{pmatrix} 10 \\ 0 \\ -3 \end{pmatrix} = \vec{u}_1$

$$= (\sin(2t)c_1 + \cos(2t)c_2) \vec{u}_1 + (-\sin(2t)c_2 + \cos(2t)c_1) \vec{v}_1 \\ \in \text{Span} \{ \vec{u}_1, \vec{v}_1 \} = E^c$$

1.3.5 $A = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$

(a) $e^A = \begin{pmatrix} e^2 & 0 \\ 0 & e^{-3} \end{pmatrix}$

(b) $A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$

Eigenvalues $\lambda_1 = 1, \lambda_2 = -1$
 $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow P = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow e^A = P \begin{pmatrix} e & 0 \\ 0 & e^{-1} \end{pmatrix} P^{-1} = \begin{pmatrix} e & e - e^{-1} \\ 0 & e^{-1} \end{pmatrix}$

(c) ~~e^A~~ $A = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$

$= \underbrace{I}_S + \underbrace{\begin{pmatrix} 0 & 0 \\ 5 & 0 \end{pmatrix}}_N$

$N^2 = 0$

S, N commute

$\Rightarrow e^A = e^S e^N = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} (I + N)$
 $= e \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$

(18)

$$(d) A = \begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix}$$

$$\lambda_1 = 2 \quad \lambda_2 = -1$$

$$\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$e^A = P \begin{pmatrix} e^2 & 0 \\ 0 & e^{-1} \end{pmatrix} P^{-1} = \begin{pmatrix} 2e^2 - e^{-1} & 2e^{-1} - 2e^2 \\ e^2 - e^{-1} & 2e^{-1} - e^2 \end{pmatrix}$$

$$e) \quad e^A = e^2 \begin{pmatrix} \cos(1) & -\sin(1) \\ \sin(1) & \cos(1) \end{pmatrix} \text{ by Corollary 3 of Sec. 1.3.}$$

$$(A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix})$$

$$(f) \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\lambda_1 = 1 \quad \lambda_2 = -1$$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$e^A = P \begin{pmatrix} e & 0 \\ 0 & e^{-1} \end{pmatrix} P^{-1} = \begin{pmatrix} \cosh(1) & \sinh(1) \\ \sinh(1) & \cosh(1) \end{pmatrix}$$

1.10.1

Let $\Phi(t)$ be a fundamental matrix solution of (2) and let $x(t) = \Phi(t)c(t)$. Then $c(0) = \Phi^{-1}(0)x_0$ and $\dot{x}(t) = \dot{\Phi}(t)c(t) + \Phi(t)\dot{c}(t) = A\Phi(t)c(t) + \Phi(t)\dot{c}(t)$ while $Ax(t) + b(t) = A\Phi(t)c(t) + b(t)$. It then follows from (1) that $\Phi(t)\dot{c}(t) = b(t)$, i.e., that $c(t) = c(0) + \int_0^t \Phi^{-1}(\tau)b(\tau)d\tau = \Phi^{-1}(0)x_0 + \int_0^t \Phi^{-1}(\tau)b(\tau)d\tau$. Thus $x(t) = \Phi(t)c(t) = \Phi(t)\Phi^{-1}(0)x_0 + \Phi(t)\int_0^t \Phi^{-1}(\tau)b(\tau)d\tau$ which is equation (3).

1.10.2

$\lambda_1 = 1, \lambda_2 = -1, v_1 = (1, 0)^T, v_2 = (1, -2)^T$ and a fundamental matrix $\Phi(t)$ with $\Phi(0) = I$ is

$$\text{given by } \Phi(t) = e^{At} = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 1/2 \\ 0 & -1/2 \end{bmatrix} = \begin{bmatrix} e^t & (e^t - e^{-t})/2 \\ 0 & e^{-t} \end{bmatrix}.$$

Note that $\Phi^{-1}(t) = \Phi(-t)$ and then

$$x(t) = \begin{bmatrix} e^t & \frac{(e^t - e^{-t})}{2} \\ 0 & e^{-t} \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{bmatrix} e^t & \frac{(e^t - e^{-t})}{2} \\ 0 & e^{-t} \end{bmatrix} \int_0^t \begin{bmatrix} e^{-\tau} & \frac{(e^{-\tau} - e^{\tau})}{2} \\ 0 & e^{\tau} \end{bmatrix} \begin{pmatrix} \tau \\ 1 \end{pmatrix} d\tau =$$

$$\begin{pmatrix} -t - 2 + \frac{5}{2}e^t + \frac{1}{2}e^{-t} \\ 1 - e^{-t} \end{pmatrix}.$$

1.10.3 (optional)

$$\dot{\Phi}(t) = \begin{bmatrix} -2e^{-2t}\cos t - e^{-2t}\sin t & -\cos t \\ -2e^{-2t}\sin t + e^{-2t}\cos t & -\sin t \end{bmatrix} = A(t)\Phi(t), \Phi^{-1}(t) = e^{2t} \begin{bmatrix} \cos t & \sin t \\ -e^{-2t}\sin t & e^{-2t}\cos t \end{bmatrix},$$

$$\Phi^{-1}(0) = I, \text{ and } x(t) = \Phi(t)\Phi^{-1}(0)x_0 + \Phi(t) \int_0^t e^{2\tau} \begin{bmatrix} \cos \tau & \sin \tau \\ -2e^{-2\tau}\sin \tau & e^{-2\tau}\cos \tau \end{bmatrix} \begin{pmatrix} 1 \\ e^{-2\tau} \end{pmatrix} d\tau =$$

$$\Phi(t) \left[x_0 + \begin{pmatrix} \frac{e^{2t}}{5}(2\cos t + \sin t) - \cos t + \frac{3}{5} \\ \frac{-e^{-2t}}{5}(2\cos t - \sin t) + \cos t - \frac{3}{5} \end{pmatrix} \right] =$$

$$\Phi(t)x_0 + \frac{1}{5} \begin{pmatrix} 2\cos^2 t - 4\sin t \cos t + 3\sin t + e^{-2t}(-5\cos^2 t + 2\sin t \cos t - \sin^2 t + 3\cos t) \\ \sin^2 t + 2\sin t \cos t + 5\cos^2 t - 3\cos t + e^{-2t}(-2\cos^2 t - 4\sin t \cos t + 3\sin t) \end{pmatrix}.$$

(2.1.1a)

$$\vec{f} = \begin{pmatrix} x_1 + x_1 x_2^2 \\ -x_2 + x_2^2 + x_1 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$$Df = \begin{pmatrix} (f_1)_{x_1} & (f_1)_{x_2} \\ (f_2)_{x_1} & (f_2)_{x_2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 + x_2^2 & 2x_1 x_2 \\ 2x_2 & -1 + 2x_2 \end{pmatrix}$$

$$\textcircled{c} D^2 f(\vec{x}_0)(\vec{x}, \vec{y})$$

$$= \sum_{j_1, j_2=1}^2 \left(\frac{\partial^2 f(\vec{x}_0)}{\partial x_{j_1} \partial x_{j_2}} \right) x_{j_1} y_{j_2}$$

$$= \begin{pmatrix} 2x_1 y_2 + 2x_2 y_1 \\ 2x_1 y_1 + 2x_2 y_2 \end{pmatrix}$$

2. 1.2

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$$(a) \vec{f}(\vec{x}) = \begin{pmatrix} -\frac{x_1}{|\vec{x}|^3} \\ -\frac{x_2}{|\vec{x}|^3} \end{pmatrix}$$

f is continuously differentiable in $\mathbb{R}^2$ except $\{\vec{0}\}$ - open set

$$(b) \vec{f}(\vec{x}) = \begin{pmatrix} |\vec{x}| + \frac{1}{x_1 - 1} \\ \sqrt{x_1 + 1} - \sqrt{x_2 + 2} \end{pmatrix}$$

not differentiable at (i) $\vec{x} = \vec{0}$

$$(ii) x_1 = 1,$$

$$(iii) x_1 \leq -1$$

$$(iv) x_2 \leq -2$$

$\Rightarrow$ Largest open set is

$$E = \{ \vec{x} \in \mathbb{R}^2 \mid x_1 > -1, x_2 > -2, x_1 \neq 1, \vec{x} \neq \vec{0} \}$$

2.1.3

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$$\dot{x} = |x|^{1/2}$$

$$x(0) = 0$$

$$\begin{array}{l} x > 0 \\ \Rightarrow \dot{x} = x^{1/2} \end{array}$$

$$\frac{\dot{x}}{x^{1/2}} = 1$$

$$x^{1/2} - x_0^{1/2} = \frac{t}{2}, \quad x_0 = 0 \Rightarrow$$

$$x = \frac{t^2}{4}$$

$$x < 0$$

$$\Rightarrow x = -\frac{t^2}{4}$$

Also we have a trivial sol $x(t) = 0$

$\Rightarrow$ 4 possible sols:

$$(1) \quad x = \begin{cases} \frac{t^2}{4}, & t \geq 0 \\ -\frac{t^2}{4}, & t \leq 0 \end{cases}$$

$$(2) \quad x = \begin{cases} 0, & t \geq 0 \\ -\frac{t^2}{4}, & t \leq 0 \end{cases}$$

$$(3) \quad x = \begin{cases} t^2/4, & t \geq 0 \\ 0, & t \leq 0 \end{cases}$$

$$(4) \quad x = 0.$$

2.2.1

(a) $u_1(t) = 1 + t$, $u_2(t) = 1 + t + t^2 + t^3/3$, $u_3(t) = 1 + t + t^2 + t^3 + 2t^4/3 + t^5/3 + t^6/9 + t^7/63$.

Mathematical induction: $u_1(t) = 1 + t$, $u_2(t) = 1 + t + t^2 + O(t^3)$ and for $n \geq 1$, assuming

$u_n(t) = 1 + t + t^2 + \dots + t^n + O(t^{n+1})$ we find that $u_{n+1}(t) = 1 + \int_0^t [1 + s + s^2 + \dots + s^n + O(s^{n+1})] ds = 1 + \int_0^t [1 + 2s + 3s^2 + \dots + (n+1)s^n + O(s^{n+1})] ds = 1 + t + t^2 + \dots + t^{n+1} + O(t^{n+2})$. QED.

(b) By separating variables and integrating we find that $x(t) = 1/(c - t)$ and the initial condition implies that $c = 1$. For $x(t) = (1 - t)^{-1}$, we have that $\dot{x}(t) = (1 - t)^{-2} = x^2(t)$ for $t \neq 0$; and since $x(0) = 1$, $t \in (-\infty, 1)$, and this function is a solution of the IVP in part (a) according to Definition 1. The Taylor series for $x(t) = 1/(1 - t) = 1 + t + \dots + t^n + \dots$, which agrees with the first $(n + 1)$ -terms in $u_n(t)$ found in part (a).

(c) $\dot{x}(t) = (3t)^{-2/3} = 1/x^2(t)$ for all $t \neq 0$; hence the function $x(t) = (3t)^{1/3}$ is a solution of the given differential equation on the interval $(-\infty, 0)$ or on the interval $(0, \infty)$. Clearly this function satisfies $x(1/3) = 1$, $1/3 \in (0, \infty)$ and hence $x(t) = (3t)^{1/3}$ is a solution of the given IVP on the interval $(0, \infty)$ according to Definition 1.

2.2.2

$u_0(t) = x_0$, $u_1(t) = x_0 + Ax_0$, $\dots$, $u_k(t) = (I + A + \dots + A^k/k!)x_0$ and $\lim_{k \rightarrow \infty} u_k(t) = e^{At}x_0$ absolutely and uniformly on any interval $[0, t_0]$.

①

Homework 02

1. Find a general solution of the following ODE:

$$ty'' - (t+1)y' + y = t^2,$$

given that $y_1(t) = e^t$ is a solution of the corresponding homogeneous equation.

use reduction of ^{order:} ~~second~~
 Look for sol. $y_2(t)$ of homogeneous problem in the form

$$y_2(t) = v(t) y_1(t)$$

$$y_2' = v' e^t + v e^t = (v y_1)'$$

$$y_2'' = (v y_1')' = v'' e^t + v' e^t + v' e^t + v e^t \\ = v'' e^t + 2v' e^t + v e^t$$

$$\Rightarrow t y_2'' - (t+1) y_2' + y_2 = 0$$

$$= t (v'' e^t + 2v' e^t + v e^t) - (t+1) (v' e^t + v e^t) + v e^t \\ = e^t [t v'' + t v' - v'] = 0$$

$$\Rightarrow t v'' + t v' - v' = 0$$

$$\text{Define } u = v' \Rightarrow t u' = (1-t)u$$

②

$$\frac{u'}{u} = \left(\frac{1}{t} - 1\right)$$

$$\int \frac{du}{u} = \int \left(\frac{1}{t} - 1\right) dt$$

$$\ln u = \ln t - t + \tilde{C}_1$$

$$\Rightarrow u = C_1 t e^{-t}$$

$$\Rightarrow v = \int u dt = C_1 \int t e^{-t} dt$$

$$= C_1 \left[-t e^{-t} + \int e^{-t} dt \right] = C_1 \left[-t e^{-t} - e^{-t} \right] + C_2$$

$$\Rightarrow y_2 = v e^t = -C_1(t+1) + C_2 e^t \quad y_1$$

$\Rightarrow$ we can take $y_2 = t+1$ as the second

solution. $\Rightarrow$ general sol. of homogeneous ODE.

$$y_h = C_1 y_1 + C_2 y_2$$

Now we can find particular nonhomogeneous sol. by method of undetermined

coefficients:

$$y_p = at^2 + bt + c$$

(3)

plug in

$$t y_p'' = (t+1) y_p' + y_p$$

$$= 2at - (t+1)(2at+b) + at^2 + b t + c = t^2$$

$$t^2 \quad \left| \begin{array}{l} -2a + a = 1 \Rightarrow a = -1 \\ 2a + 2a - b + b = 0 \end{array} \right.$$

$$t \quad \left| \begin{array}{l} 2a + 2a - b + b = 0 \end{array} \right.$$

$$t^0 \quad \left| \begin{array}{l} -b + c = 0 \Rightarrow b = c \end{array} \right. \quad \text{We can choose } b \text{ arbitrary}$$

because $y_2 = t+1$,

E.g. choose $b = -a \Rightarrow c = 0$

$$y_p = -t^2$$

General sol: $y = c_1 e^t + c_2(t+1) + (-t^2)$

Alternative way is to solve through variation of parameters:

Wronskian $w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^t & (t+1) \\ e^t & 1 \end{vmatrix}$

$$= e^t - (t+1)e^t = -te^t \neq 0 \text{ for } t \neq 0$$

$$\Rightarrow y = c_1 y_1 + c_2 y_2 + y_2 \int \frac{y_1(t^2)}{w} dt - y_1 \int \frac{y_2(t^2)}{w} dt$$

(4)

$$= c_1 y_1 + c_2 y_2 + (t+1) \int \frac{e^x + t}{-t e^t} dt - e^t \int \frac{(t+1)t}{-t e^t} dt$$

$$= c_1 y_1 + c_2 y_2 + (t+1)(-t) + e^t \int (t+1)e^{-t} dt$$

$$= c_1 y_1 + c_2 y_2 + (t+1)(-t) + e^t \bar{e}^{-t} (-t - t - 1)$$

$$= c_1 y_1 + c_2 y_2 - t^2 - t - t - 1$$

$$= c_1 y_1 + c_2 y_2 - t^2 - 2(t+1)$$

can be absorbed into y_2

$$= \tilde{c}_1 e^t + \tilde{c}_2 (t+1) - t^2$$

- the same as previous.