

math 5/2/462

HW 03 Solutions

(1)

2.1.9

$$\dot{x} = x^{-3}$$

$$x(0) = 2$$

$$\left(\frac{1}{x^2}\right)' = -2$$

$$\frac{1}{x^2} - \frac{1}{x(0)^2} = -2t$$

$$x = \pm \frac{1}{\sqrt{-2t + \frac{1}{x(0)^2}}} = \pm \frac{1}{\sqrt{-2t + \frac{1}{4}}}$$

$\Rightarrow x(0)$ choose + sign

$$x = \frac{2}{\sqrt{1-8t}}$$

for $-\infty < t < \frac{1}{8}$

for $t \rightarrow 2^-$ $x(t) \rightarrow \infty$

(2.1.5)

$$\dot{x} = \frac{1}{2x}$$

$$x(1) = 1$$

$$(\dot{x}^2) = 1$$

$$x = \pm \sqrt{t} + C$$

$$x(1) = 1$$

$$\Rightarrow x = \sqrt{t} \quad - \quad \text{is a sol. at } (0, \infty)$$

$$\text{But for } t \rightarrow 0 \quad \dot{x}(t) = \frac{1}{2\sqrt{t}} \rightarrow \infty$$

$\Rightarrow$ undefined at $t=0$.
 because $\dot{x}(0)$ does not exist.

(3)

2.2.4 $A \in \mathbb{R}^{n \times n}$, $A(t) \in C([-a_0, a_0])$

Pf Define $\Phi_0(t) = I \in \mathbb{R}^{n \times n}$ identity matrix

$$\Phi_{k+1}(t) = I + \int_0^t A(s) \Phi_k(s) ds \quad (*)$$

Since $A(t) \in C([-a_0, a_0]) \Rightarrow \|A(t)\| \leq M \forall t \in [-a_0, a_0]$

$$\text{True } 0 < a < a_0 \Rightarrow [-a, a] \subset [-a_0, a_0]$$

$$\text{True } t \in [-a, a]$$

$$\Rightarrow \Phi_1(t) = I + \int_0^t A(s) ds \Rightarrow \|\Phi_1 - \underbrace{\Phi_0}_I\|$$

$$\leq \int_0^{|t|} \|A(s)\| ds \leq Ma$$

By induction assuming that $\|\Phi_k - \Phi_{k-1}\| \leq Ma^k \forall t \in [-a, a]$

$$\Rightarrow \|\Phi_{k+1} - \Phi_k\| \leq \left\| \int_0^t A(s) \Phi_k(s) ds - \int_0^t A(s) \Phi_{k-1}(s) ds \right\|$$

$$\leq \int_0^{|t|} \|A(s)\| \|\Phi_k - \Phi_{k-1}\| ds \leq (Ma)^k Ma = (Ma)^{k+1}$$

$$\Rightarrow \text{True } \forall k = 1, 2, \dots$$

Define $\alpha = Ma$ and take $m > n > N$

(4)

$$\begin{aligned} \Rightarrow \|\Phi_m - \Phi_n\| &\leq \sum_{k=n}^{m-1} \|\Phi_{k+1} - \Phi_k\| \\ &\leq \sum_{k=N}^{\infty} \|\Phi_{k+1} - \Phi_k\| \leq \sum_{k=N}^{\infty} (M\alpha)^k = \frac{\alpha^N}{1-\alpha} \xrightarrow{N \rightarrow \infty} 0 \\ &\text{provided } \alpha < 1 \end{aligned}$$

$\Rightarrow \{\Phi_n\}$ is a Cauchy sequence

$\Rightarrow$ converges to continuous function uniformly on $t \in [-a, a]$, where we choose $a = \min(\frac{1}{M}, a_0)$.

Taking limit $n \rightarrow \infty$ on both sides of (*):

$$\Phi = I + \int_0^t A(s) \Phi(s) ds$$

Φ, A continuous $\Rightarrow \int_0^t A(s) \Phi(s) ds$ is differentiable w.r.t $\Phi = A\Phi$ with $\Phi(0) = I$
 $t \in [-a, a]$

Now to prove uniqueness:

Assume that Φ, Ψ are solutions of IVP

$$\Rightarrow \Phi, \Psi \in C([-a, a])$$

$\Rightarrow \|\Phi - \Psi\|$ has maximum on $[-a, a]$, at some $t_1 \in [-a, a]$

$$\Rightarrow \max_{t \in [-a, a]} \|\Phi(t) - \Psi(t)\| = \|\Phi(t_1) - \Psi(t_1)\|$$

(5)

$$= \left\| \int_0^{t_1} A(s) (\Phi(s) - \Psi(s)) ds \right\|$$

$$\leq M \int_0^{t_1} \|\Phi(s) - \Psi(s)\| ds \leq M a \|\Phi(t_1) - \Psi(t_1)\|$$

But $Ma < 1 \Rightarrow$ inequality is only true for

$$\|\Phi(t_1) - \Psi(t_1)\| = 0$$

$$\Rightarrow \Phi(t) = \Psi(t) \quad \square$$

2.2.5 (optional)

By the lemma in this section, f is locally Lipschitz in E . Therefore, given $x_0 \in E$, there exists a $K_0 > 0$ and an $\varepsilon > 0$ such that $N_\varepsilon(x_0) \subset E$ and for all $x, y \in N_\varepsilon(x_0)$, $|f(x) - f(y)| \leq K_0 |x - y|$. Next, $T \circ u(t)$ is continuous at $t = 0$. Therefore for $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|t| < \delta$ then $|T \circ u(t) - T \circ u(0)| = |T \circ u(t) - x_0| < \varepsilon$. Choose $a > 0$ such that $a < \min(\delta, 1/K_0)$. Then for $I = [-a, a]$, $t \in I$ and $u, v \in V \equiv \{u \in C(I) \mid \|u - x_0\| \leq \varepsilon\}$.

$$|T \circ u(t) - T \circ v(t)| = \left| \int_0^t [f(u(s)) - f(v(s))] ds \right| \leq \int_0^t |f(u(s)) - f(v(s))| ds \leq c \|u - v\|$$

where $c = K_0 a < 1$. Thus, by the contraction mapping principle, there exists a unique $u(t) \in V \subset C(I)$ such that $T \circ u(t) = u(t)$ for all $t \in I$.

7.2(6)

If $x(t)$ is a continuous function on I that satisfies the integral equation, then $x(0) = x_0$ and $\dot{x}(t) = \frac{d}{dt} \int_0^t f(x(s)) ds = f(x(t))$ for all $t \in I$ by the fundamental theorem of calculus since $f(x(t)) \in C(I)$; and therefore $x(t)$ is differentiable and it satisfies the initial value problem (2) for all $t \in I$. Conversely, if $x(t)$ is a solution of the initial value problem (2) for all $t \in I$, then $x(t)$ is differentiable and hence continuous on I and $x(t) \in E$ for all $t \in I$; therefore,

$x'(t) = f(x(t))$ implies that $x(t) = \int_0^t f(x(s)) ds + c$ for all $t \in I$ and clearly $c = x(0) = x_0$. Thus, $x(t)$ satisfies the integral equation for all $t \in I$.

7.2(9)

Suppose that there is a constant $K_0 > 0$ such that for all $x, y \in E$, $|f(x) - f(y)| \leq K_0 |x - y|$.

Then, given $\varepsilon > 0$, choose $\delta = \varepsilon/K_0 > 0$ to get that for $x, y \in E$ with $|x - y| < \delta$,

$|f(x) - f(y)| \leq K_0 |x - y| < K_0 \delta = \varepsilon$. Therefore, f is uniformly continuous on E .

7.2(10) (Optional)

(a)

Follow the hint for $\delta < 1$; and for $\delta \geq 1$, choose $x = 1$ and $y = 1/3$ to show that

$$|f(x) - f(y)| = 2 > 1 = \varepsilon.$$

(b)

Use the result of part (a) and Problem 9 to show that $f(x) = 1/x$ does not satisfy a Lipschitz condition on $(0, 1)$.

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2.3.1

$$\begin{aligned}\vec{\dot{x}} &= A \vec{x} \\ \vec{x}(0) &= \vec{y}\end{aligned}$$

From the fundamental thm for lin system: $\vec{u}(t, \vec{y}) = e^{At} \vec{y}$.

$$\Phi(t) = \frac{\partial \vec{u}}{\partial \vec{y}} = e^{At} I = e^{At}$$

$$\dot{\Phi} = A e^{At} = A \Phi$$

$$\Phi(0) = I$$

2.3.2

$$\vec{\dot{x}} = \vec{f}(\vec{x})$$

$$\vec{x}(0) = \vec{y}$$

$$\vec{f}(\vec{x}) = \begin{pmatrix} -x_1 \\ -x_2 + x_1^2 \\ x_3 + x_1^2 \end{pmatrix}$$

(a) $\dot{x}_1 = -x_1 \Rightarrow u_1(t, \vec{y}) = y_1 e^{-t}$

$\dot{x}_2 = -x_2 + x_1^2 = -x_2 + y_1^2 e^{-2t}$
- inhomogeneous lin. ODE.

$$\Rightarrow u_2(t, \vec{y}) = -\frac{y_1^2}{2} e^{-2t} + \left(\frac{y_1^2}{2} + y_2\right) e^{-t}$$

$$u_3(t, \vec{y}) = -\frac{y_1^2}{3} e^{-2t} + \left(\frac{y_1^2}{3} + y_3\right) e^{-t}$$

8

$$\dot{\Phi}(t) = \frac{\partial \tilde{u}(t, \vec{y})}{\partial \vec{y}} = \frac{\partial u_i}{\partial y_j}$$

$$= \begin{pmatrix} e^{-t} & 0 & 0 \\ -2y_1 \bar{e}^{-2t} - \bar{e}^{-t} & \bar{e}^{-t} & 0 \\ \frac{2}{3} y_1 (\bar{e}^{-t} - \bar{e}^{-2t}) & 0 & e^{-t} \end{pmatrix}$$

$$Df(t, \vec{y}) = \begin{pmatrix} -1 & 0 & 0 \\ 2y_1 \bar{e}^{-t} & -1 & 0 \\ 2y_1 \bar{e}^{-t} & 0 & 1 \end{pmatrix}$$

$$Df \cdot \Phi(t, \vec{y}) = \begin{pmatrix} -e^{-t} & 0 & 0 \\ 2y_1 (2\bar{e}^{-2t} - \bar{e}^{-t}) & -\bar{e}^{-t} & 0 \\ \frac{2}{3} y_1 (e^{-t} + 2\bar{e}^{-2t}) & 0 & e^{-t} \end{pmatrix}$$

$$\dot{\Phi}(t, \vec{y}) = \begin{pmatrix} -e^{-t} & 0 & 0 \\ 2y_1 (2\bar{e}^{-2t} - \bar{e}^{-t}) & -\bar{e}^{-t} & 0 \\ \frac{2}{3} y_1 (e^{-t} + 2\bar{e}^{-2t}) & 0 & e^{-t} \end{pmatrix}$$

$$\Rightarrow \dot{\Phi} = Df \Phi$$

$$(b) \quad \dot{\vec{x}} = \begin{pmatrix} x_1^2 \\ x_2 + \frac{1}{x_1} \end{pmatrix}, \quad \vec{x}(0) = \vec{y}$$

④

$$\dot{x}_1 = x_1^2$$

$$\Rightarrow -\frac{1}{x_1} + \frac{1}{x_1(0)} = t, \quad x_1(0) = y_1$$

$$u_1(t) = \frac{y_1}{1-y_1 t}$$

$$\dot{x}_2 = x_2 + \frac{1}{x_1} = x_2 + \frac{1-y_1 t}{y_1}$$

$$u_2 = x_2 = y_2 e^t + \int_0^t e^{t-t'} \cdot \frac{1-y_1 t'}{y_1} dt'$$

$$= y_2 e^t + \left(-\frac{1-e^t}{y_1} \right) \cancel{1 + t e^t} + 1 + t e^t$$

$$\Phi = \frac{\partial u_i}{\partial y_j} = \begin{pmatrix} \frac{1}{(1-y_1 t)^2} & 0 \\ \frac{1-e^t}{y_1^2} & e^t \end{pmatrix}$$

$$D\vec{f} = \begin{pmatrix} 2x_1 & 0 \\ -\frac{1}{x_1^2} & 1 \end{pmatrix} = \begin{pmatrix} \frac{2y_1}{1-y_1 t} & 0 \\ -\frac{(1-y_1 t)^2}{y_1^2} & 1 \end{pmatrix}$$

$$Df\Phi = \begin{pmatrix} \frac{2y_1}{(1-y_1 t)^3} & 0 \\ -\frac{e^t}{y_1^2} & e^t \end{pmatrix}$$

$$\dot{\Phi} = \begin{pmatrix} \frac{2y_1}{(1-y_1)^2} & 0 \\ -\frac{e^t}{y_1^2} & e^t \end{pmatrix} = D\vec{f}\Phi$$

(6)

(2.3.5) By the corollary of Sect. 7.3

and from Liouville's thm:

$$\det \frac{\partial \vec{u}}{\partial \vec{y}}(t, \vec{x}_0) = e^{\int_0^t \text{trace } D\vec{f}[u(s, \vec{x}_0)] ds}$$

$$= e^{\int_0^t \nabla \cdot \vec{f}(u(s, \vec{x}_0)) ds} \quad \text{since } D\vec{f} = \nabla \cdot \vec{f}$$

$$= \sum_{i=1}^n \frac{\partial}{\partial x_i} f_i$$

2.3.6

(11)

$\vec{f} \in C^1(E)$, $E \subset \mathbb{R}^n$ open,
 $\vec{x}_0 \in E$.

$\vec{u}(t, \vec{y}_0)$ is the unique sol
 of $\frac{d\vec{x}}{dt} = \vec{f}(\vec{x})$

$\vec{x}(0) = \vec{y}_0$ for $t \in [0, a]$

$\vec{y} = \vec{u}(t, \vec{y}_0)$ is volume preserving

iff $J(\vec{y}) = \det \frac{\partial \vec{u}}{\partial \vec{y}} = \text{constant}$.

but $\left. \frac{\partial \vec{u}}{\partial \vec{y}} \right|_{\vec{y}=\vec{y}_0} = I \Rightarrow \text{Constant} = 1$

From Problem 2.3.5 it follows

that $e^{\int_0^t \nabla \cdot \vec{f}(\vec{u}(s, \vec{y}_0)) ds} = 1 \quad \forall t \in [0, a]$

$\Rightarrow \int_0^t \nabla \cdot \vec{f}(\vec{u}(s, \vec{y}_0)) ds = 0 \quad \forall t \in [0, a]$
 and $\vec{y}_0 \in E$.

$\Leftrightarrow$ by continuity $\nabla \cdot \vec{f}(\vec{x}) = 0 \quad \forall \vec{x} \in E$. □

2.4.1

(12)

(a) $\dot{x} = x^2$

$x(0) = x_0$

$-\frac{1}{x} + \frac{1}{x_0} = t$

$\Rightarrow x(t) = \frac{x_0}{(1 - x_0 t)}$, $t \in (-\infty, \frac{1}{x_0})$
 $= (\alpha, \beta)$
for $x_0 > 0$

lim and $(\alpha, \beta) = (\frac{1}{x}, \infty)$ for $x_0 < 0$
for $x_0 = 0 \Rightarrow (\alpha, \beta) = (-\infty, \infty)$

$\Rightarrow$ $x_0 > 0$ $\lim_{t \rightarrow \infty} x(t) = 0$

$\lim_{t \rightarrow \frac{1}{x_0}^-} x(t) = +\infty$

$x_0 < 0$ $\lim_{t \rightarrow \frac{1}{x_0}^+} x(t) = -\infty$

$\lim_{t \rightarrow +\infty} x(t) = 0$

$x_0 = 0$ $(-\infty, \infty) = (\alpha, \beta)$

$x(t) = 0$

$\lim_{x \rightarrow \pm\infty} x(t) = 0$

⑧

$$\dot{x} = \sec x$$

$$x(0) = 0$$

$$\int_{x_0}^x dx \cos x = t$$

$x_0 = 0$

$$\Rightarrow \sin x = t$$

$$x = \arcsin t$$

$$(\alpha, \beta) = (-1, 1) = E$$

$$\lim_{t \rightarrow -1^+} x(t) = -\frac{\pi}{2} \in \partial E$$

$$\lim_{t \rightarrow 1^-} x(t) = \frac{\pi}{2} \in \partial E$$

⑬

2.4.2b

$$\dot{x}_1 = \frac{1}{2x_1}, \quad x_1(0) = 1 \Rightarrow x_1 = \sqrt{1+t}$$

$$\dot{x}_2 = x_2^2, \quad x_2(0) = 1 \Rightarrow x_2 = \frac{1}{1-t}$$

$$(\alpha, \beta) = (-1, 1)$$

$$\vec{x}(t) \rightarrow \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \text{ as } t \rightarrow -1^+$$

$$E = \{x_1 > 0\}$$

$$|\vec{x}(t)| \rightarrow \infty \text{ as } t \rightarrow 1^-$$