

HW 07 Solutions

math 762/512

①

2.4.3

Assume $\beta < \infty$. If $\lim_{t \rightarrow \beta^-} x(t)$ does not exist, then there exists a sequence $t_n \rightarrow \beta^-$ such that

$\{x(t_n)\}$ is not Cauchy; i.e., there exists an $\varepsilon > 0$ such that for all integers N , there exist

integers $n > m \geq N$ such that $|x(t_n) - x(t_m)| \geq \varepsilon$. Thus, for $N = 1$, there exist integers

$n_1 > m_1 \geq 1$ such that $|x(t_{n_1}) - x(t_{m_1})| \geq \varepsilon$; for $N = n_1$, there exist integers $n_2 > m_2 \geq n_1$

such that $|x(t_{n_2}) - x(t_{m_2})| \geq \varepsilon$; $\dots$ for $N = n_j$, there exist integers $n_{j+1} > m_{j+1} \geq n_j$ such that

$|x(t_{n_j}) - x(t_{m_j})| \geq \varepsilon$. Hence, the arc length of $\Gamma_+ \geq \sum_{n=1}^{\infty} |x(t_{n+1}) - x(t_n)| \geq \sum_{j=1}^{\infty} |x(t_{n_j}) - x(t_{m_j})| \geq$

$\sum_{j=1}^{\infty} \varepsilon = \infty$. Hence if $\beta < \infty$ and the arc length of Γ_+ is finite, it follows that $\lim_{t \rightarrow \beta^-} x(t)$ exists.

2.4.5

2

Suppose $\lim_{t \rightarrow \beta^-} x(t) = x_1 \in E$. Then since E is open, there is an $\varepsilon > 0$ such that $N_{2\varepsilon}(x_1) \subset E$ and $\overline{N_\varepsilon(x_1)} \subset E$. Assume that $\beta < \infty$. Then there is a $\delta > 0$ such that for $|t - \beta| < \delta$, $|x(t) - x_1| < \varepsilon$. Since $x(t)$ is continuous and $[0, \beta - \delta]$ is a compact set, $K = \{y \in \mathbb{R}^n \mid y = x(t), t \in [0, \beta - \delta]\} \cup \{y \in \mathbb{R}^n \mid |y - x_1| \leq \varepsilon\}$ is a compact subset of E ; furthermore,

$\Gamma_+ \subset K$. Thus, by Corollary 2, β is not finite; i.e., $\beta = \infty$. Next, we show that $f(x_1) = 0$.

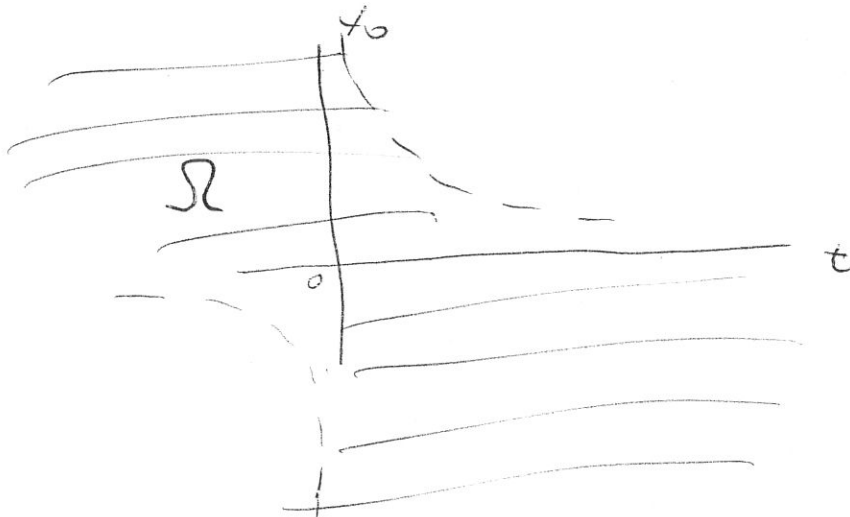
Suppose that $f(x_1) \neq 0$, say $|f(x_1)| = \delta > 0$. Then by the continuity of f , there exists an $\varepsilon > 0$ such that $|x - x_1| < \varepsilon$ implies that $x \in E$ and $|f(x)| \geq \delta/2$. Since $x(t) \rightarrow x_1$ and $\dot{x}(t) \rightarrow v_1 \equiv f(x_1)$ as $t \rightarrow \infty$, it follows that for this $\varepsilon > 0$, there exists a $t_0 \geq 0$ such that for all $t \geq t_0$, $|x(t) - x_1| < \varepsilon$ and $|\dot{x}(t) - v_1| < \varepsilon$, i.e., for all $t \geq t_0$, $|\dot{x}(t)| = |f(x(t))| \geq \delta/2$ and $|v_1 \cdot \dot{x}(t)| = |v_1| |\dot{x}(t)| \cdot |\cos \theta_1(t)| \geq |v_1| \delta/4$ where $\theta_1(t)$ is the angle between $\dot{x}(t)$ and v_1 and $|\cos \theta_1(t)| \geq 1/2$ for all $t \geq t_0$. Then by the mean value theorem, for all $t > t_0$, there is a $\tilde{t} \in (t_0, t)$ such that $v_1 \cdot x(t) - v_1 \cdot x(t_0) = (t - t_0) v_1 \cdot \dot{x}(\tilde{t})$; thus, $|v_1| |x(t) - x(t_0)| \geq |v_1| [x(t) - x(t_0)] = |t - t_0| |v_1 \cdot \dot{x}(\tilde{t})| \geq |t - t_0| |v_1| \delta/4$ and therefore $|x(t) - x(t_0)| \geq |t - t_0| \delta/4 \geq 2\varepsilon$ for $t \geq t_0 + 8\varepsilon/\delta$. Since $|x(t_0)| < \varepsilon$, this implies that for $t \geq t_0 + 8\varepsilon/\delta$, $|x(t) - x_1| \geq |x(t) - x(t_0)| - |x(t_0) - x_1| \geq 2\varepsilon - \varepsilon = \varepsilon$, a contradiction since $|x(t) - x_1| < \varepsilon$ for all $t \geq t_0$. Thus, $f(x_1) = 0$; and x_1 is an equilibrium point of (1), i.e., $x(t) = x_1$ is the solution of (1) satisfying the initial condition $x(0) = x_1$.

2.5.1

$$\dot{x} = x^2$$

$$x(0) = x_0$$

$$\Rightarrow x(t) = \frac{x_0}{1 - x_0 t}$$

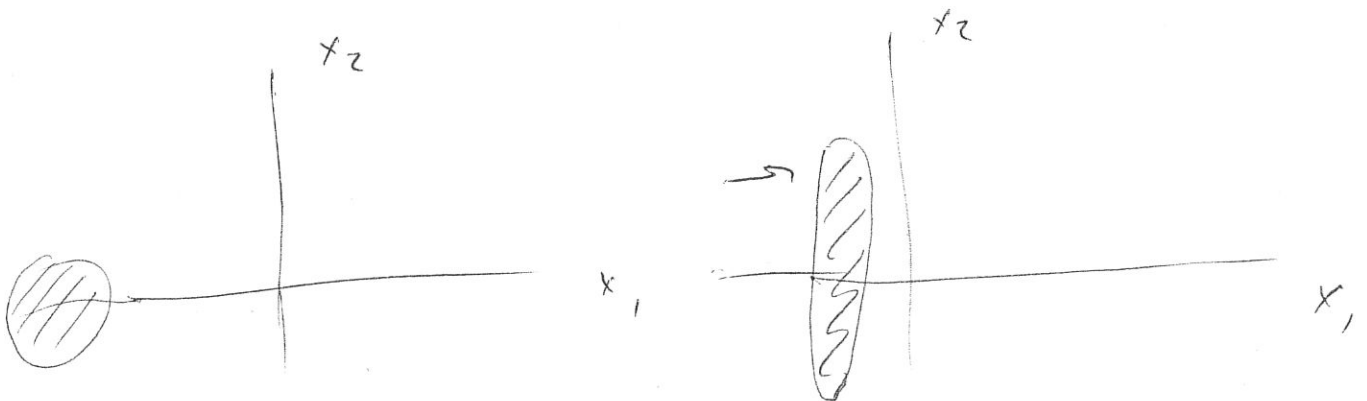


2.5.3

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\dot{\vec{x}} = A \vec{x}$$

$$\Rightarrow \Phi_t(\vec{x}) = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix} \vec{x}$$



(1)

2.5.6

$$\vec{f}(\vec{x}) = \begin{pmatrix} -x_1 \\ -x_2 + x_1^2 \\ x_3 + x_1^2 \end{pmatrix} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix}$$

$$\vec{x}(0) = \vec{y}$$

$$\Rightarrow x_1 = y_1 e^{-t}$$

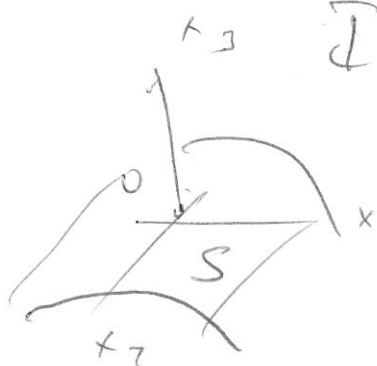
$$x_2' = -x_2 + x_1^2 \Rightarrow x_2 = y_2 e^{-t} + \int_0^t e^{-t+s} y_1^2 e^{-2s} ds$$

$$= y_2 e^{-t} + y_1^2 (e^{-t} - e^{-2t})$$

$$x_3 = y_3 e^t - \frac{y_1^2}{3} e^{-2t} + \frac{y_1^2}{3} e^t$$

$$\Rightarrow \phi_t(\vec{y}) = \begin{pmatrix} y_1 e^{-t} \\ y_2 e^{-t} + y_1^2 (e^{-t} - e^{-2t}) \\ y_3 e^t - \frac{y_1^2}{3} e^{-2t} + \frac{y_1^2}{3} e^t \end{pmatrix}$$

$$\text{If } y_3 = -\frac{y_1^2}{3} \Rightarrow \phi_t(\vec{y}) = \begin{pmatrix} y_1 e^{-t} \\ y_2 e^{-t} + y_1^2 (e^{-t} - e^{-2t}) \\ -\frac{y_1^2}{3} e^t \end{pmatrix} \in S$$



2.5.7

2

$$\dot{x} = \frac{1}{x^2}$$

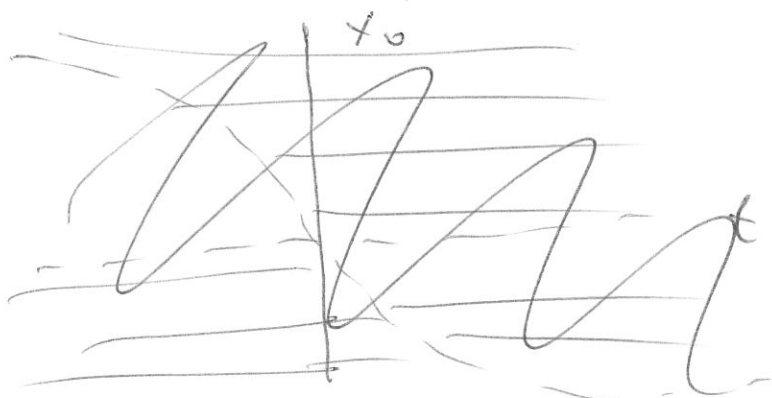
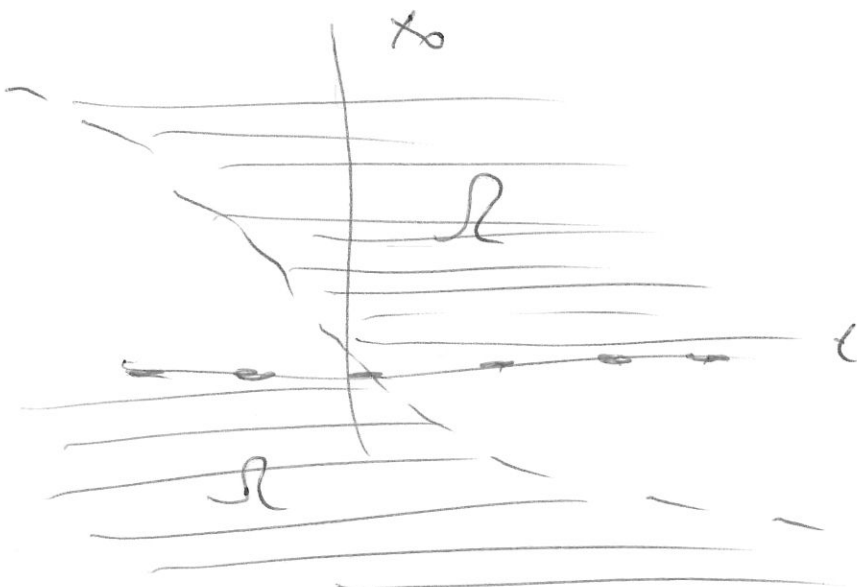
$$x^2 dx = 1$$

$$\frac{x^3}{3} - \frac{x_0^3}{3} = t$$

$$x = (3t + x_0^3)^{1/3}$$

$$\text{For } x_0 > 0 \Rightarrow (\alpha, \beta) = (-\frac{x_0^3}{3}, \infty)$$

$$\phi_t(x_0) \rightarrow 0 \in \partial E \text{ as } t \rightarrow (-\frac{x_0^3}{3})^+$$



2.6.1

$$(a) \quad \vec{f} = \begin{pmatrix} x_1 - x_1 x_2 \\ x_2 - x_1^2 \end{pmatrix} = \begin{pmatrix} x_1(1-x_2) \\ x_2 - x_1^2 \end{pmatrix} \quad (1)$$

$$\vec{f} = \vec{0} \quad \text{either } x_1 = 0 \Rightarrow x_2 = 0 \quad \left| \frac{\partial f_i}{\partial x_j} = \begin{pmatrix} 1-x_2 & -x_1 \\ -2x_1 & 1 \end{pmatrix} \right.$$

$$\text{or } x_2 = 1 \Rightarrow x_1 = \pm 1$$

Equilibrium point

$$\vec{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$D\vec{f}(\vec{0}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \text{source (unstable node)} \\ \lambda = 1$$

E. p.

$$\vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$D\vec{f}(\begin{pmatrix} 1 \\ 1 \end{pmatrix}) = \begin{pmatrix} 0 & -1 \\ -2 & 1 \end{pmatrix}$$

$$\lambda = 2, -1$$

$\Rightarrow$ saddle

$$\text{E. p. } \vec{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow D\vec{f}(\begin{pmatrix} -1 \\ 1 \end{pmatrix}) = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$$

$$\lambda = 2, -1$$

$\Rightarrow$ saddle

$$(b) \quad \vec{f} = \begin{pmatrix} -4x_2 + 2x_1x_2 - 8 \\ 4x_2^2 - x_1^2 \end{pmatrix}$$

②

$$\frac{\partial f_i}{\partial x_j} = D\vec{f} = \begin{pmatrix} 2x_2 & -4 + 2x_1 \\ -2x_1 & 8x_2 \end{pmatrix}$$

$$\vec{f} = 0 \Rightarrow x_1 = \pm 2x_2$$

$$\Rightarrow -4x_2 \pm 4x_2^2 - 8 = 0$$

$$\Rightarrow x_2 = -1 \quad \text{or} \quad x_2 = 2 \quad \text{for } x_1 = 2x_2 \text{ only}$$

$$\text{Ex. } \vec{x} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} \Rightarrow D\vec{f} = \begin{pmatrix} -2 & -8 \\ 2 & -8 \end{pmatrix}$$

$$\lambda = -5 \pm i\sqrt{2}$$

$\Rightarrow$ ~~sink~~ ^{sink} (unstable focus)

$$\text{Ex. } \vec{x} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \Rightarrow D\vec{f} = \begin{pmatrix} 4 & 4 \\ -8 & 16 \end{pmatrix}$$

$$\lambda = 12, 8$$

$\Rightarrow$ source (unstable node)

(3)

$$(c) \quad \vec{f} = \begin{pmatrix} 2x_1 - 2x_1 x_2 \\ 2x_2 - x_1^2 + x_2^2 \end{pmatrix}$$

$$\frac{\partial f_i}{\partial x_j} = D\vec{f} = \begin{pmatrix} 2 - 2x_2 & -2x_1 \\ -2x_1 & 2 + 2x_2 \end{pmatrix}$$

$$\vec{f} = \vec{0} \Rightarrow x_1 = 0 \text{ or } x_2 = +1$$

$$\Downarrow$$

$$2x_2 + x_2^2 = 0$$

$$x_2 = 0 \text{ or } x_2 = -2$$

$$\Downarrow$$
~~$$2x_2 - 1 + x_2^2 = 0$$~~

$$+2 - x_1^2 + 1 = 0$$

$$x_1 = \pm \sqrt{3}$$

Ex. p $\vec{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow D\vec{f}|_0 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$
 $\lambda = 2$

Ex. r $\vec{x} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \Rightarrow D\vec{f}|_{\vec{x}} = \begin{pmatrix} 6 & 0 \\ 0 & -2 \end{pmatrix}$
 $\lambda = 6, -2$

saddle

Ex. r $\vec{x} = \begin{pmatrix} \pm\sqrt{3} \\ 1 \end{pmatrix} \Rightarrow D\vec{f}|_{\vec{x}} = \begin{pmatrix} 0 & \pm 2\sqrt{3} \\ \pm 2\sqrt{3} & 4 \end{pmatrix}$

$$\lambda = 6, -2$$

saddle

$$(d) \quad \vec{f} = \begin{pmatrix} -x_1 \\ -x_2 + x_1^2 \\ x_3 + x_1^2 \end{pmatrix}$$

$$\frac{\partial f_i}{\partial x_j} = D\vec{f} = \begin{pmatrix} -1 & 0 & 0 \\ 2x_1 & -1 & 0 \\ 2x_1 & 0 & 1 \end{pmatrix}$$

$$\vec{f} = \vec{0} \Rightarrow x_1 = 0 = x_2 = x_3$$

$$\text{E.p. } \vec{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow D\vec{f}(\vec{0}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\lambda = \pm 1 \quad \underline{\text{saddle}}$$

(e) see sol. 2.6.2 with $k \rightarrow \mu$.

2.6.2

5

$$\dot{\vec{x}} = \vec{f}(\vec{x}), \mu > 0$$

$$\vec{f}(\vec{x}) = \begin{pmatrix} x_2 - x_1 \\ \mu x_1 - x_2 - x_1 x_3 \\ x_1 \cdot x_2 - x_3 \end{pmatrix}$$

$$\vec{f}(\vec{x}) = \vec{0}$$

$$\Rightarrow x_1 = x_2$$

$$\left. \begin{aligned} \mu x_1 - x_2 - x_1 x_3 &= \mu x_1 - x_1 - x_1 x_3 = 0 \\ x_1^2 - x_3 &= 0 \end{aligned} \right\}$$

$$\Rightarrow \mu x_1 - x_1 - x_1^3 = 0$$

$$\Rightarrow \vec{x} = \vec{0} \quad \text{or} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \pm \sqrt{\mu-1} \\ \pm \sqrt{\mu-1} \\ \mu-1 \end{pmatrix}$$

(1) $0 < \mu \leq 1$ - unique equilibrium point $\vec{x} = \vec{0}$

(2) $\mu > 1 \Rightarrow 3$ equilibrium points
 $\vec{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \vec{x} = \begin{pmatrix} \sqrt{\mu-1} \\ \sqrt{\mu-1} \\ \mu-1 \end{pmatrix}, \vec{x} = \begin{pmatrix} -\sqrt{\mu-1} \\ -\sqrt{\mu-1} \\ \mu-1 \end{pmatrix}$

$$A = Df(\vec{x}) = \begin{pmatrix} -1 & 1 & 0 \\ \mu - x_3 & -1 & -x_1 \\ x_2 & x_1 & -1 \end{pmatrix}$$

⑥

$$\det(A - \lambda I) = 0 \Rightarrow (\lambda + 1)^3 + (\lambda + 1)(x_1^2 + x_3 - \mu) + x_1 x_2 = 0$$

$$\underline{\vec{x} = \vec{0}} \Rightarrow (\lambda + 1)^3 - (\lambda + 1)\mu = 0$$

$$\lambda_1 = -1, \quad \lambda_2 = -1 + \sqrt{\mu}, \quad \lambda_3 = -1 - \sqrt{\mu}$$

$$\Rightarrow \vec{x} = \vec{0} \quad \begin{cases} \text{sink if } 0 < \mu < 1 \\ \text{nonhyperbolic eq. point if } \mu = 1 \\ \text{saddle if } \mu > 1 \end{cases}$$

$$\vec{x} = \begin{pmatrix} \pm \sqrt{\mu - 1} \\ \pm \sqrt{\mu - 1} \\ \mu - 1 \end{pmatrix} \Rightarrow \mu > 1$$

$$\Rightarrow \lambda_1 = -2, \quad \lambda_{2,3} = \frac{-1 \pm \sqrt{5 - 4\mu}}{2}$$

$$\text{If } \mu > \frac{5}{4} \Rightarrow \lambda_{2,3} = \frac{-1 \pm i\sqrt{4\mu - 5}}{2},$$

$$\Rightarrow \operatorname{Re} \lambda_{2,3} = -\frac{1}{2} < 0 \Rightarrow \underline{\text{sink}}$$

$$\text{If } 1 < \mu < \frac{5}{4} \Rightarrow \lambda_{2,1} < 0 \Rightarrow \underline{\text{sink}}$$

④

2.6.3

$$H: \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

$$H(\vec{x}) = \begin{pmatrix} x_1 \\ x_2 + x_1^2 \\ x_3 + \frac{x_1^2}{3} \end{pmatrix}$$

the inverse $\vec{y} = H(\vec{x})$

$$\Rightarrow y_1 = x_1$$

$$y_2 = x_2 + x_1^2 = x_2 + y_1^2 \Rightarrow x_2 = y_2 - y_1^2$$

$$y_3 = x_3 + \frac{x_1^2}{3} = x_3 + \frac{y_1^2}{3} \Rightarrow x_3 = y_3 - \frac{y_1^2}{3}$$

$$\Rightarrow H^{-1}(\vec{y}) = \begin{pmatrix} y_1 \\ y_2 - y_1^2 \\ y_3 - \frac{y_1^2}{3} \end{pmatrix} \quad \text{continuous in } \mathbb{R}^3$$

$$\vec{f}(\vec{x}) = \begin{pmatrix} -x_1 \\ -x_2 + x_1^2 \\ x_3 + x_1^2 \end{pmatrix}$$

$$Df(\vec{x}) = \begin{pmatrix} -1 & 0 & 0 \\ 2x_1 & -1 & 0 \\ 2x_1 & 0 & 1 \end{pmatrix} \Rightarrow \det Df(\vec{x}) = 1 \neq 0$$

$$A = Df(\vec{0}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(8)

Assume that $\vec{y} = H(\vec{x})$

$$\dot{\vec{y}} = H(\vec{x}) = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 + 2x_1 \dot{x}_1 \\ \dot{x}_3 + 2x_1 \dot{x}_1 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -x_2 - x_1^2 + 2x_1(-x_1) \\ x_3 + x_1^2 + \frac{2}{3}(x_1(-x_1)) \end{pmatrix}$$

$$= \begin{pmatrix} -x_1 \\ -x_2 - x_1^2 \\ x_3 + \frac{x_1^2}{3} \end{pmatrix} = \begin{pmatrix} -y_1 \\ -y_2 \\ y_3 \end{pmatrix}$$

$$\Rightarrow \dot{\vec{y}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{y} = Df(\vec{0}) \vec{y}$$

$$\Rightarrow \dot{\vec{y}} = A \vec{y}$$