

math 462/512

HW 5 Solutions

①

2.7.1

$$\textcircled{1} \quad \dot{x}_1 = x_1 + 6x_2 + x_1x_2$$

$$\dot{x}_2 = 4x_1 + 3x_2 - x_1^2$$

$$A = \begin{pmatrix} 1 & 6 \\ 4 & 3 \end{pmatrix}$$

$$\frac{\lambda_1 = -3}{\vec{v}_1 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}} \quad \frac{\lambda_2 = 7}{\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

$$P = (\vec{v}_1 \vec{v}_2) = \begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix} \quad P^{-1} = \frac{1}{5} \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$$

$$\vec{y} = P^{-1} \vec{x}$$

$$\Rightarrow B = P^{-1} A P = \begin{pmatrix} -3 & 0 \\ 0 & 7 \end{pmatrix}$$

~~$$G = P^{-1} F P = P^{-1} \begin{pmatrix} x_1 & x_2 \\ -x_1^2 \end{pmatrix}$$~~

$$\vec{x} = P \vec{y}$$

$$x_1 = 3y_1 + y_2$$

$$x_2 = -2y_1 + y_2$$

$$G = P^{-1} F = P^{-1} \begin{pmatrix} x_1 & x_2 \\ -x_1^2 \end{pmatrix} = P^{-1} \begin{pmatrix} (3y_1 + y_2)(-2y_1 + y_2) \\ -(3y_1 + y_2)^2 \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} (3y_1 + y_2)(-2y_1 + y_2) + (3y_1 + y_2)^2 \\ 2(3y_1 + y_2)(-2y_1 + y_2) + 3(3y_1 + y_2)^2 \end{pmatrix}$$

(2)

$$= \frac{(3y_1 + y_2)}{5} \begin{pmatrix} -2y_1 + y_2 + 3y_1 + y_2 \\ 2(-2y_1 + y_2) + 3(3y_1 + y_2) \end{pmatrix}$$

$$= \frac{3y_1 + y_2}{5} \begin{pmatrix} y_1 + 2y_2 \\ -13y_1 - y_2 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 3y_1^2 + 7y_1y_2 + 2y_2^2 \\ -39y_1^2 - 16y_1y_2 - y_2^2 \end{pmatrix}$$

(2.7.2)

$$\dot{x}_1 = -x_1$$

$$\dot{x}_2 = x_2 + x_1^2$$

$$A = B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\vec{F} = \vec{G} = \begin{pmatrix} 0 \\ x_1^2 \end{pmatrix}$$

$$U = \begin{pmatrix} e^{-t} & 0 \\ 0 & 0 \end{pmatrix}$$

$$V = \begin{pmatrix} 0 & 0 \\ 0 & e^t \end{pmatrix}$$

$$\vec{u}(t) = U(t) \vec{a} = \begin{pmatrix} e^{-t} a_1 \\ 0 \end{pmatrix}$$

$$\vec{u}(t) = U(t) \vec{a} + \int_0^t \begin{pmatrix} 0 & 0 \\ e^{-(t-s)} & 0 \end{pmatrix} \vec{F}(s) ds = \int_0^t \begin{pmatrix} 0 & 0 \\ 0 & e^{-(t-s)} \end{pmatrix} \begin{pmatrix} 0 \\ s^2 \end{pmatrix} ds$$

$$= \begin{pmatrix} e^{-t} a_1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} 0 \\ 0 \end{pmatrix} ds + \int_t^\infty \begin{pmatrix} 0 \\ e^{t-s-2s} a_1^2 \end{pmatrix} ds \quad (3)$$

$$= \begin{pmatrix} e^{-t} a_1 \\ -e^{-\frac{2t}{3}} a_1^2 \end{pmatrix}$$

$$\begin{aligned} \vec{u}^{(1)} &= \begin{pmatrix} e^{-t} a_1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} e^{-(t-s)} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ e^{-2s} a_1^2 \end{pmatrix} ds \\ &\quad - \int_t^\infty \begin{pmatrix} 0 & 0 \\ 0 & e^{t-s} \end{pmatrix} \begin{pmatrix} 0 \\ e^{-2s} a_1^2 \end{pmatrix} ds = \begin{pmatrix} e^{-t} a_1 \\ -e^{-\frac{2t}{3}} a_1^2 \end{pmatrix} = \vec{u} \end{aligned}$$

$$\Rightarrow \vec{u}^{(j+1)} = \vec{u}^{(j)} \text{ for } j \geq 2.$$

$$\Rightarrow \vec{u}(t, \vec{x}) = \vec{u}^{(1)}(t, \vec{x}) = \begin{pmatrix} e^{-t} a_1 \\ -e^{-\frac{2t}{3}} \frac{a_1^2}{3} \end{pmatrix}$$

$$x_2 = -\frac{a_1^2}{3}$$

$$\Rightarrow S: x_2 = -\frac{x_1^2}{3}$$

For unstable manifold u :
 $t \rightarrow -t$

$$\begin{aligned} \dot{x}_1 &= x_1 \\ \dot{x}_2 &= -x_2 - x_1^2 \end{aligned}$$

~~At $t \rightarrow -t$~~

But we need to interchange: $x_1 \leftrightarrow x_2$:

(7)

$$\dot{x}_2 = -x_2 - x_1^2$$

$$\dot{x}_1 = x_1$$

$$\begin{pmatrix} \dot{x}_2 \\ \dot{x}_1 \end{pmatrix} = \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}_{A=B} \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} + \underbrace{\begin{pmatrix} -x_1^2 \\ 0 \end{pmatrix}}_{\vec{F} = \vec{G}}$$

$$U = \begin{pmatrix} e^{-t} & 0 \\ 0 & 0 \end{pmatrix} \quad V = \begin{pmatrix} 0 & 0 \\ 0 & e^t \end{pmatrix}, \quad \vec{a} = \begin{pmatrix} a_2 \\ a_1 \end{pmatrix}$$

$$\vec{u}^{(1)} = U \vec{a} = \begin{pmatrix} a_2 e^{-t} \\ 0 \end{pmatrix}$$

$$\vec{u}^{(2)} = \begin{pmatrix} a_2 e^{-t} \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} e^{-t+s} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -a_1 \\ 0 \end{pmatrix} ds$$

$$- \int_t^\infty \begin{pmatrix} 0 & 0 \\ 0 & e^{t-s} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} ds = \vec{u}^{(2)}$$

$$\vec{u}^{(1)} = \vec{u}^{(2)} = \dots \Rightarrow \vec{u} = \begin{pmatrix} a_2 e^{-t} \\ 0 \end{pmatrix}$$

$$\psi_1 = u_1(0, a_1, 0) = 0$$

$$\Rightarrow u_1 x_1 = 0$$

2.7.4

⑤

$$\begin{cases} \dot{x}_1 = -x_1 \\ \dot{x}_2 = -x_2 + x_1^2 \\ \dot{x}_3 = x_3 + x_2^2 \end{cases}$$

$$A = B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\vec{F} = \vec{G} = \begin{pmatrix} 0 \\ x_1^2 \\ x_2^2 \end{pmatrix}$$

$$U = \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^t \end{pmatrix}$$

$$V = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e^t \end{pmatrix}$$

$$\vec{u}^{(1)} = U \vec{a} = \begin{pmatrix} e^{-t} a_1 \\ e^{-t} a_2 \\ 0 \end{pmatrix}$$

$$\vec{u}^{(2)} = \begin{pmatrix} e^{-t} a_1 \\ e^{-t} a_2 \\ 0 \end{pmatrix} + \int_0^t U(t-s) \vec{G}(\vec{u}^{(1)}(s)) ds$$

$$- \int_0^\infty V(t-s) \vec{G}(\vec{u}^{(1)}(s)) ds$$

$$= \begin{pmatrix} e^{-t} a_1 \\ e^{-t} a_2 \\ 0 \end{pmatrix} \neq \int_0^t \begin{pmatrix} 0 \\ e^{-(t-s)} a_2 e^{-2s} \\ 0 \end{pmatrix} ds$$

(6)

$$- \int_t^{\infty} \begin{pmatrix} 0 \\ 0 \\ e^{t-s} e^{-2s} a_2^2 \end{pmatrix} ds$$

$$= \begin{pmatrix} e^{-t} a_1 \\ e^{-t} a_2 - e^{-2t} a_1^2 + e^{-t} a_1^2 \\ -\frac{1}{3} e^{-2t} a_2^2 \end{pmatrix}$$

$$u^{(3)} = \begin{pmatrix} e^{-t} a_1 \\ e^{-t} a_2 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} 0 \\ e^{-(t-s)} e^{-2s} a_1^2 \\ 0 \end{pmatrix} ds$$

$$- \int_t^{\infty} \begin{pmatrix} 0 \\ 0 \\ e^{t-s} (e^{-s} a_2 - e^{-2s} a_1^2 + e^{-s} a_1^2)^2 \end{pmatrix} ds$$

$$= \begin{pmatrix} e^{-t} a_1 \\ e^{-t} a_2 - e^{-s-t} a_1^2 \Big|_0^t \\ - \int_t^{\infty} e^{t-s} \left[(a_2 + a_1^2)^2 + a_1^4 e^{-2s} - 2a_1^2 (a_2 + a_1^2) e^{-s} \right] ds \end{pmatrix}$$

$$= \begin{pmatrix} e^{-t} a_1 \\ e^{-t} (a_2 + a_1^2) - e^{-2t} a_1^2 \\ -\frac{e^{-4t}}{5} a_1^4 + e^{-3t} a_1^2 \frac{(a_2 + a_1^2)}{2} + \frac{1}{3} e^{-2t} \frac{(a_2 + a_1^2)^2}{3} \end{pmatrix}$$

$$= u^{(4)} = \dots u^{(j)}, \quad j \geq 3.$$

(7)

$$\Rightarrow \psi(a_1, a_2) = u_3 \begin{pmatrix} 0, a_1, a_2, 0 \end{pmatrix}$$

$$= -a_2^2/3 - a_1^2 a_2 - \frac{a_1^4}{30}$$

$$\Rightarrow S: x_3 = -x_2^2/3 - x_1^2 x_2 - \frac{x_1^4}{30}$$

To find u : $t \rightarrow -t$:

$$\vec{y} = \begin{pmatrix} +1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ -x_1^2 \\ -x_2^2 \end{pmatrix}$$

To write in standard form we need to interchange :

$$\begin{pmatrix} x_3 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_3 \\ x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -x_2^2 \\ 0 \\ -x_1^2 \end{pmatrix}$$

$A=B$

$F=G$

$$u = \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad v = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{pmatrix}$$

$$\vec{a} = \begin{pmatrix} a_3 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{u}^{(1)} = u \vec{a} = \begin{pmatrix} e^{-t} a_3 \\ 0 \\ 0 \end{pmatrix} \quad (8)$$

$$\vec{u}^{(2)} = \begin{pmatrix} e^{-t} a_3 \\ 0 \\ 0 \end{pmatrix} + \int_0^\infty \begin{pmatrix} e^{-t+s} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} ds - \int_t^\infty \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} ds$$

$$= \vec{u}^{(1)} = \dots \vec{u}^{(j)}, j \geq 1$$

$$\Rightarrow \vec{u} = \begin{pmatrix} e^{-t} a_3 \\ 0 \\ 0 \end{pmatrix}$$

$$u_2 = 0$$

$$u_3 = 0$$

$$\Rightarrow u: x_1 = x_2 = 0, \text{ i.e. } x_3\text{-axis.}$$

$$(2.7.5) \begin{cases} \dot{x}_1 = -x_1 \\ \dot{x}_2 = -x_2 + x_1^2 \\ \dot{x}_3 = x_3 + x_2^2 \end{cases} \Rightarrow x_1 = c_1 e^{-t}$$

$$\Rightarrow x_2 = -c_1^2 e^{-2t} + (c_2 + c_1^2) e^{-t}$$

$$x_2 = -c_1^4 e^{4t/5} + c_1^2 (c_2 + c_1^2) \frac{e^{-3t}}{2}$$

$$- \frac{(c_2 + c_1^2)^2}{3} e^{-2t} + \frac{(30c_3 + c_1^4 + 5c_1^2 c_2 + 10c_2^2)}{30} e^t$$

$$\lim_{t \rightarrow \infty} \Phi_t(\vec{c}) = \vec{0} \text{ if } 30c_3 + c_1^4 + 5c_1^2 c_2 + 10c_2^2 = 0$$

$$\Rightarrow S: x_3 = -\frac{x_2^2}{3} - \frac{x_1^2 x_2}{6} + \frac{x_1^4}{30}$$

$$\lim_{t \rightarrow \infty} \Phi_t(\vec{c}) = \vec{0} \text{ iff } c_1 = c_2 = 0$$

(9)

$$\Rightarrow \mathcal{U}: x_1 = x_2 = 0$$

(2.7.6) Since $\vec{F} \in C^1(E)$

$$\Rightarrow \forall \varepsilon > 0 \exists \delta: \forall \vec{z} \in N_\delta(0), \|\mathcal{D}\vec{F}(\vec{z}) - \mathcal{D}\vec{F}(0)\|$$

$$= \|\mathcal{D}\vec{F}(\vec{z})\| < \varepsilon \Rightarrow \forall \vec{x}, \vec{y} \in N_\delta(0),$$

$$\|\vec{F}(\vec{x}) - \vec{F}(\vec{y})\| \leq \|\mathcal{D}\vec{F}(\vec{z})\| \cdot \|\vec{x} - \vec{y}\| < \varepsilon \|\vec{x} - \vec{y}\| \quad \square$$

7

2.9.2

$$(a) \quad \vec{f} = \begin{pmatrix} x_1^2 - x_2^2 - 1 \\ 2x_2 \end{pmatrix} = \vec{0}$$

$$\Rightarrow \text{equilibrium points } \vec{x}_{01} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \vec{x}_{02} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$Df(\vec{x}) = A = \begin{pmatrix} 2x_1 & -2x_2 \\ 0 & 2 \end{pmatrix}$$

$$A(\vec{x}_{01}) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow \text{unstable proper node}$$

$$A(\vec{x}_{02}) = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} - \text{saddle (unstable)}$$

$$(b) \quad \vec{f} = \begin{pmatrix} x_2 - x_1^2 + 2 \\ 2x_2^2 - 2x_1x_2 \end{pmatrix} = \vec{0} \Rightarrow \begin{matrix} x_2 = 0, x_1 = \pm\sqrt{2} \\ \text{or } x_1 = x_2 \\ \Rightarrow x_1 = x_2 = -1, 2 \end{matrix}$$

$$Df(\vec{x}) = A = \begin{pmatrix} -2x_1 & 1 \\ -2x_2 & 4x_2 - 2x_1 \end{pmatrix}$$

$$A\left(\begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix}\right) = \begin{pmatrix} -2\sqrt{2} & 1 \\ 0 & 2\sqrt{2} \end{pmatrix} \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \text{general eigenvector} \\ \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\lambda_1 = \lambda_2 = -2\sqrt{2}$$

Sol. of linearized syst:

(2)

$$\vec{x} = c_1 e^{-\sqrt{2}t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-2\sqrt{2}t} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]$$

$\Rightarrow$ asymptotically stable proper node.

$$\textcircled{2} A \begin{pmatrix} -\sqrt{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 2\sqrt{2} & 1 \\ 0 & 2\sqrt{2} \end{pmatrix} \quad d_1 = d_2 = 2\sqrt{2}$$

$\Rightarrow$ unstable node

$$A \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 2 & -2 \end{pmatrix} \quad \lambda = \pm \sqrt{2} \Rightarrow \text{saddle (unstable)}$$

$$A \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 & 1 \\ -4 & 4 \end{pmatrix} \quad \lambda = \pm 2\sqrt{2} \Rightarrow \text{saddle, (unstable)}$$

$$\textcircled{3} \vec{f} = \begin{pmatrix} -4x_1 - 2x_2 + 4 \\ x_1 x_2 \end{pmatrix}$$

$$\textcircled{2} \text{ fix } \vec{x}_{01} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$\vec{x}_{02} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$A = D \vec{f}(\vec{x}) = \begin{pmatrix} -4 & -2 \\ x_2 & x_1 \end{pmatrix}$$

(3)

~~A(0,0)~~

$\vec{x}_{01}$

$$A \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 & -2 \\ 2 & 0 \end{pmatrix} \quad \lambda = -2$$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$(A + 2I)^2 \vec{v}_2 = \vec{0} \Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\vec{x} = c_1 \vec{v}_1 e^{-2t} + c_2 e^{-2t} (\vec{v}_2 + t \vec{v}_1)$$

asymptotically stable node.

$\vec{x}_{02}$

$$A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 & -2 \\ 0 & 1 \end{pmatrix} \quad \begin{matrix} \lambda = -4 \\ \lambda = 1 \end{matrix}$$

$\Rightarrow$ saddle (unstable)

(2.9.4)

$$V = x_1^2 + x_2^2$$

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \dot{\vec{x}} = A\vec{x} + \vec{f}$$

(a)

$$\vec{f} = \begin{pmatrix} -x_1^3 - x_1 x_2^2 \\ -x_2^3 - x_2 x_1^2 \end{pmatrix}$$

$$\begin{aligned} \dot{V} &= 2x_1 \dot{x}_1 + 2x_2 \dot{x}_2 = 2x_1 (-x_2^3 - x_1^3 - x_1 x_2^2) \\ &\quad + 2x_2 (x_1^3 - x_2^3 - x_2 x_1^2) \end{aligned}$$

$$= -2x_1^4 - 2x_1^2 x_2^2 - 2x_2^4 - 2x_2^2 x_1^2 < 0 \quad \forall \vec{x} \neq 0 \quad (4)$$

$\Rightarrow \vec{0}$ asymptotically stable

$$(b) \quad \vec{f} = \begin{pmatrix} x_1^3 + x_1 x_2^2 \\ x_2^3 + x_2 x_1^2 \end{pmatrix}$$

$$\dot{V} = 2x_1(-x_2 + x_1^3 + x_1 x_2^2) + 2x_2(x_1 + x_2^3 + x_2 x_1^2)$$

$$= 2x_1^4 + 2x_1^2 x_2^2 + 2x_2^4 + 2x_2^2 x_1^2 > 0 \quad \forall \vec{x} \neq 0$$

$\Rightarrow \vec{0}$ unstable

$$(c) \quad \vec{f} = \begin{pmatrix} -x_1 x_2 \\ x_1^2 \end{pmatrix}$$

$$\dot{V} = 2x_1(-x_2 - x_1 x_2) + 2x_2(x_1 + x_1^2)$$

$$= -2x_1^2 x_2 + 2x_2 x_1^2 = 0 \quad \forall \vec{x}$$

$\Rightarrow \vec{0}$ is stable but not asymptotically stable.

Solution curves - circles around origin.

2.9.5

(b) Take

$$\begin{cases} \dot{x}_1 = x_1 - 3x_2 + x_1^2 \\ \dot{x}_2 = -x_1 + x_2 - x_2^2 \end{cases}$$

(5)

$$u = -x_1 x_2$$

$$\begin{aligned} \dot{u} &= -(x_1 - 3x_2 + x_1^2)x_2 \\ &\quad - x_1(-x_1 + x_2 - x_2^2) \end{aligned}$$

$$= -2x_1 x_2 + 3x_2^2 + x_1^2 + O(|\vec{x}|^3)$$

$$= (x_1 - x_2)^2 + 2x_2^2 + O(|\vec{x}|^3) > 0 \text{ for}$$

small enough δ : $|\vec{x}| < \delta$, $\vec{x} \neq 0$

$\Rightarrow \vec{0}$ unstable by stronger criterion

of instability, leave e.g. for $x_1 = -x_2$
 $u > 0$, $\vec{x} \neq 0$

Instead - look at linearization.

$$A = \begin{pmatrix} 1 & -3 \\ -1 & 1 \end{pmatrix} \quad \lambda = 1 \pm \sqrt{3} \text{ - saddle}$$

$\Rightarrow \vec{0}$ unstable.

Second equilibrium point

$$\vec{x}_{02} = \begin{pmatrix} -0.59\dots \\ -0.94\dots \end{pmatrix}$$

- skip.

2.9.7

1

$$\dot{x}' + f(x) \dot{x} + g(x) = 0$$

Take $x_1 = x$, $\dot{x}_2 = -g(x)$

$$\ddot{x} = \ddot{x}_1 = -g(x_1) - f(x_1) \dot{x}_1$$

$$= \ddot{x}_2 - F'(x_1) \dot{x}_1 = \frac{d}{dt} (x_2 - F(x_1))$$

$$\Rightarrow \dot{x}_1 = x_2 - F(x_1) \quad (\text{up to arbitrary constant})$$

$$F(x_1) = \int_0^{x_1} f(s) ds$$

Take $V(x) = \frac{x_2^2}{2} + R(x_1)$, $G(x, t) = \int_0^{x_1} g(s) ds$

$$\Rightarrow \dot{V} = x_2 \dot{x}_2 + g(x_1) \dot{x}_1$$

$$= x_2 (-g(x_1)) + g(x_1) (x_2 - F(x_1))$$

$$= -g(x_1) F(x_1)$$

$$\Rightarrow \text{if } g F < 0 \Rightarrow \dot{V} < 0, x \neq 0$$

$$\text{Also } V > 0 \text{ if } G > 0, x \neq 0$$

$$\Rightarrow \text{Liapunov function } \dot{V} < 0 \Rightarrow x=0 \text{ is unstable equilibrium point.}$$

$$\text{If } gF > 0$$

(2.9)

$\Rightarrow \dot{V} < 0 \Rightarrow$ asymptotically stable equilibrium point.

$$(2.9.8) \quad \dot{x} + \varepsilon(x^2 - 1)\dot{x} + x = 0$$

From (2.9.7) : $f = \varepsilon(x^2 - 1)$

$$g = x$$

$$F = \int_x^0 \varepsilon(x'^2 - 1) dx' = \varepsilon \frac{x^3}{3} - \varepsilon x < 0 \text{ for } 0 < |x| < \sqrt{3}$$

$$G = \int_0^x x' dx' = \frac{x^2}{2} > 0 \text{ for } x \neq 0$$

$$\Rightarrow gF = \varepsilon \frac{x^4}{3} - \varepsilon x^2 < 0 \text{ for } 0 < |x| < \sqrt{3}$$

$\Rightarrow$ (origin is unstable)



Problem 1 solution

(1)

$$\dot{x} = y$$

$$\Rightarrow \begin{cases} \dot{x} = y \\ \dot{y} = x - y \sin y \end{cases} \quad (1)$$

Try $u = xy$

$$\Rightarrow \dot{u} = x^2 + y^2 - xy \sin y$$

$$\dot{u}(0,0) = 0$$

$$\text{for } x, y \rightarrow 0 \Rightarrow \dot{u} = x^2 + y^2 + O(x^2 y)$$

$$\Rightarrow \dot{u} > 0 \text{ for } |x|, |y| < \delta \text{ for}$$

small enough $\delta \Rightarrow$ positive definite.

$$u(x, x) = x^2 > 0 \quad \forall x \neq 0$$

$\Rightarrow$ by the stronger criterion of instability (1) is unstable.

2

problem 2 solution.

$$\begin{cases} \dot{x} = y^3 + x^2 y \\ \dot{y} = x^3 - x y^2 \end{cases} \quad (1)$$

Try $u = xy$

$$\dot{u} = \dot{x}y + x\dot{y} = (y^3 + x^2 y)y + x(x^3 - xy^2)$$

$$= x^4 + y^4 > 0 \text{ for } \vec{x} \neq 0$$

$$\dot{u}(0,0) = 0$$

$$u(x,x) = x^2 > 0 \quad \forall x \neq 0$$

by the stronger criterion of
 $\Rightarrow$ instability (1) is unstable.