Solution of the system of linear ODEs with constant coefficients by a matrix exponent

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Solution of initial value problem (IVP)

$$\dot{\mathbf{x}} = A\mathbf{x},\ \mathbf{x}|_{t=0} = \mathbf{x}_0,\ \mathbf{x} \in \mathbb{R}^n,\ A \in \mathbb{R}^{n\times n} \tag{1}$$

can be written as a matrix exponent

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0. \tag{2}$$

To calculate e^{At} we can first find $e^{At}\mathbf{w}$, where $\mathbf{w}$ is the generalized complex eigenvector of A for the eigenvalue λ determined by the equation $(A-\lambda I)^k\mathbf{w} = 0$, where k is the positive integer and $I \in \mathbb{R}^{n\times n}$ is the identity matrix. Because I commutes with any $n \times n$ matrix, we decompose A as $A = (A-\lambda I) + \lambda I$ and write

$$e^{At} = e^{t(A-\lambda I)+t\lambda I} = e^{t\lambda I}e^{t(A-\lambda I)} = e^{\lambda t}e^{t(A-\lambda I)}. \tag{3}$$

But $e^{t(A-\lambda I)}\mathbf{w}$ for the generalized eigenvector $\mathbf{w}$ is truncated to the finite number of terms according to the definition of $\mathbf{w}$ as follows

$$e^{t(A-\lambda I)}\mathbf{w} = \left[I + t(A-\lambda I) + \frac{t^2(A-\lambda I)^2}{2!}\dots\right]\mathbf{w} = \sum_{l=0}^{k-1}\frac{t^l(A-\lambda I)^l}{l!}\mathbf{w}, \tag{4}$$

where k is smaller of equal to the algebraic multiplicity of the eigenvalue λ of the matrix A.

The equation (3) and (4) allows to write the particular solution $\mathbf{z}_\lambda(t)$ of (2) as follows

$$\mathbf{z}_\lambda(t) = e^{\lambda t}e^{t(A-\lambda I)}\mathbf{w} = e^{\lambda t}\left[I + t(A-\lambda I) + \ldots + \frac{t^{k-1}(A-\lambda I)^{k-1}}{(k-1)!}\right]\mathbf{w} = e^{\lambda t}\sum_{l=0}^{k-1}\frac{t^l(A-\lambda I)^l}{l!}\mathbf{w}. \tag{5}$$

We choose any basis of the generalized eigenvectors of A in $\mathbb{R}^n$ as $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p, \mathbf{w}_{p+1}, \bar{\mathbf{w}}_{p+1}, \ldots \mathbf{w}_m, \bar{\mathbf{w}}_m\}$ with $n = k+2(m-k)$. Here $\mathbf{u}_1, \ldots, \mathbf{u}_p$ are the generalized eigenvectors of A for the real eigenvalues $\lambda_1, \ldots, \lambda_p$, respectively. Also $\mathbf{w}_{p+1}, \bar{\mathbf{w}}_{p+1}, \ldots, \mathbf{w}_m, \bar{\mathbf{w}}_m$ are the generalized eigenvectors for the complex eigenvalues $\lambda_{p+1}, \bar{\lambda}_{p+1}, \ldots, \lambda_m, \bar{\lambda}_m$, respectively. For that basis we obtain n linearly independent solutions of $\dot{\mathbf{x}} = A\mathbf{x}$ according to (5). We form the following fundamental matrix from these solutions:

$$F(t) = \left[\mathbf{z}_{\lambda_1}, \ldots, \mathbf{z}_{\lambda_p}, Im(\mathbf{z}_{\lambda_{p+1}}), Re(\mathbf{z}_{\lambda_{p+1}}) \ldots Im(\mathbf{z}_{\lambda_m}), Re(\mathbf{z}_{\lambda_m})\right] \tag{6}$$

so that the general solution is given by $F\mathbf{c}_0$, with $\mathbf{c}_0 \in \mathbb{R}^n$ being the arbitrary vector of constant coefficients. Solution of IVP (1) is then given by

$$\mathbf{x}(t) = F(t)F_0^{-1}\mathbf{x}_0, \quad F_0 = F(0). \tag{7}$$