Reduction of order

Assume that we know 1 sol $y = y_1$

of DE: $y'' + p(t)y' + q(t)y = 0$ (*)

$\Rightarrow$ we can find second sol if

we assume $y = v(t)\, y_1(t)$ and

plug in into eqn. (*)

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$$\Rightarrow \quad y' = (v y_1)' = v' y_1 + v y_1'$$

$$y'' = (v' y_1 + v y_1')'$$

$$= v'' y_1 + 2 v' y_1' + v y_1''$$

Substitute into (*)!

$$(v'' y_1 + 2 v' y_1' + v y_1'') + p (v' y_1 + v y_1')$$

$$+ q v y_1$$

$$= y_1 v'' + (2 y_1' + p y_1) v' + \underbrace{(y_1'' + p y_1' + q y_1)}_{= 0 \text{ because of } (*)} v = 0$$

$$= y_1 v'' + (2 y_1' + p y_1) v' = 0$$

- 1st order eq for $u = v'$:

$$y_1 u' = -(2 y_1' + p y_1) u$$

$$\int \frac{du}{u} = - \int \frac{(2 y_1' + p y_1)}{y_1} \, dt \Rightarrow \text{find } u = v'$$

after that $v = \int u \, dt$

Easy to work on particular examples:

Ex ~~[illegible]~~ find second sol.: ~~of given DE~~.

$t^2 y'' + 2t y' - 2y = 0$, $t > 0$, $y_1(t) = t$.

$y_2 = t\,v(t)$

$\Rightarrow \quad y_2' = t v' + v$

$y_2'' = \cancel{t v'' + 2v'}$ $\quad t v'' + v' + v' = t v'' + 2v'$

$\Rightarrow \quad t^2 [t v'' + 2v'] + 2t [t v' + v] - 2tv = 0$

$\Rightarrow \quad t^3 v'' + 4t^2 v' = 0$

– this linear 1 order eqn for $w = v'$. So we reduced order.

$\Rightarrow \quad t^3 w' + 4t^2 w = 0$

$t w' + 4w = 0$

$\frac{dw}{w} = -\frac{4}{t}\,dt$

$\ln w = c - 4 \ln t$

$w = \frac{c_1}{t^4} = v'$

$v = -\frac{1}{3} c_1 t^{-3} + c_2 = \hat{c}_1 t^{-3} + c_2$

$\Rightarrow \quad y = \hat{c}_1 t^{-2} + c_2 t \Rightarrow$ second sol. t^{-2}.