thm Stronger criterion for instability

Assume $\vec{x}_0 = 0$, $\dot{\vec{x}} = \vec{f}(\vec{x})$, $\vec{f}(\vec{0}) = \vec{0}$.

$\vec{f} \in C^1(E_1)$, E_1 open $\subset \mathbb{R}^n$.

Define a function $\mathcal{U}(\vec{x}) : E \to \mathbb{R}$, $E = \{ \vec{x} \in \mathbb{R}^n \mid |\vec{x}| \le K \}$

for $K > 0$, $E \subset E_1$.

$\mathcal{U}(\vec{x})$ has the following properties:

(1) $\mathcal{U}(\vec{x}) \in C^1(E)$

(2) $\mathcal{U}(\vec{0}) = 0$

(3) $\dot{\mathcal{U}} > 0 \;\; \forall \vec{x} \in E \setminus \vec{0}$, $\dot{\mathcal{U}}(\vec{0}) = 0$

(4) $\forall \delta > 0 : \exists\, \vec{x} \in N_\delta(\vec{0})$ such that $\mathcal{U}(\vec{x}) > 0$

$\Rightarrow$ $\vec{0}$ is unstable.

Pf that — From (4): $\forall \delta$ s.t. $0 < \delta < K$, $\exists\, \vec{x}_\delta$ such

$0 < |\vec{x}_\delta| < \delta$ and $\mathcal{U}(\vec{x}_\delta) > 0$.

Assume that $\vec{\Phi}_t(\vec{x}_\delta)$ is the flow.

Then we cannot have $\lim_{t\to\infty} \vec{\Phi}_t(\vec{x}_\delta) = 0$

because $\mathcal{U}(0) = 0 < \mathcal{U}(\vec{x}_\delta)$ while $\dot{\mathcal{U}}(\vec{x}) > 0$ for $\vec{x} \ne \vec{0}$ (from (3))

Since $\dot{\mathcal{U}} > 0$ for $\vec{x} \ne \vec{0}$ by (3) and $\mathcal{U}$ is

continuous by (1) and since $\vec{\Phi}_t(\vec{x}_\delta)$ is bounded away from the origin (by continuity)

$\Rightarrow \exists\, m > 0 : \dot{\mathcal{U}}\left(x_t(\vec{x}_\delta)\right) \geq m > 0$ for $t \geq 0$

provided $|\Phi_t(\vec{x}_\delta)| \leq K$.

But $\quad \mathcal{U}\left(\Phi_t(\vec{x}_\delta)\right) - \mathcal{U}(\vec{x}_\delta) = \int_0^t \dot{\mathcal{U}}\left(\vec{\Phi}_s(\vec{x}_\delta)\right) ds$

$\geq m t \quad (*)$

$\Rightarrow$ But $\mathcal{U}(\vec{x})$ is continuous

$\Rightarrow$ is bounded in E. But as $t \to \infty$

$(*)$ implies that $\vec{\Phi}_t(\vec{x}_\delta)$ reaches

boundary: $\partial E = \{|\vec{x}| = K\}$.

$\Rightarrow \forall\, \varepsilon > 0$, $0 < \varepsilon < K$, and $\forall\, \delta > 0$ but small,

$\exists\, \vec{\Phi}_t(\vec{x}_\delta) : |\vec{\Phi}_0(\vec{x}_\delta)| < \delta$ but $|\vec{\Phi}_t(\vec{x}_\delta)| > \varepsilon$

for some $t \Rightarrow$ instability $\square$

Ex.

$$\dot{\vec{x}} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\vec{x} + \begin{pmatrix} x_2^3 \\ x_1^2 \end{pmatrix} \qquad \vec{x} \in \mathbb{R}^2$$

$\mathcal{U} = x_2^2 - x_1^2$ – not sign definite

$$\dot{\mathcal{U}} = 2x_2\dot{x}_2 - 2x_1\dot{x}_1$$

$$= 2x_2(x_2 + x_1^2) - 2x_1(-x_1 + x_2^3)$$

$$= 2(x_1^2 + x_2^2) + 2x_1^2x_2 - 2x_1x_2^3$$

$$= 2(x_1^2 + x_2^2) + \underbrace{2x_1x_2(x_1 - x_2^2)}_{\to 0 \text{ for } |\vec{x}| \to 0} > 0$$

$\Rightarrow$ $\vec{0}$ is unstable by above thm.