Evaluation of definite integrals from dumbbell contours

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We consider the definite integrals of the β-function type

$$I = \int_0^1 \left(\frac{x}{1-x}\right)^{\alpha} R(x)dx, \qquad -1 < \alpha < 1, \tag{1}$$

where $R(x)$ is the rational function such that it does not have poles at the closed interval $x \in [0,1]$ and

$$R(x) \to const \text{ for } x \to \infty. \tag{2}$$

To evaluate (1) we extend its integrand into the complex plane $z \in \mathbb{C}$ as follows

$$f(z) := \left(\frac{z}{1-z}\right)^{\alpha} R(z) \tag{3}$$

and define a branch cut at the segment of the real line $[0,1]$ which connects branch point $z = 0$ and $z = 1$ of $f(z)$. Then we choose the branch of $f(z)$ such that

$$f(x + i0) = f(x) > 0 \qquad \text{for} \qquad 0 < x < 1. \tag{4}$$

Here and below $x + i0$ and $x - i0$ means the limit $\epsilon \to 0^+$, $\epsilon > 0$ for $x + i\epsilon$ and $x - i\epsilon$, respectively.

To obtain $f(x - i0),\ 0 < x < 1$ we move from $x + i0,\ 0 < x < 1$ to $x - i0,\ 0 < x < 1$ either around the branch point $z = 0$ in the counterclockwise (positive) direction on the angle 2π thus adding $2\pi\alpha$ to the argument of $f(z)$ from z^{α} factor in (3) or around the branch point $z = 1$ in the clockwise (negative) direction on the angle -2π thus adding $-2\pi(-\alpha) = 2\pi\alpha$ to the argument of $f(z)$ from $(1-z)^{-\alpha}$ factor in (3). Thus in both cases

$$f(x - i0) = e^{i2\pi\alpha} f(x + i0) \qquad \text{for} \qquad 0 < x < 1. \tag{5}$$

It also proves that $f(z)$ is analytic in $\mathbb{C} \setminus [0,1]$.

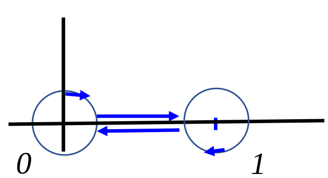

FIG. 1. Dumbbell contour.

We integrate over a dumbbell contour shown in Fig. 1 consisting of the line segments $L_1 :\ [1 - \rho - i0, \rho - i0]$, $L_2 :\ [\rho + i0, \rho + i0]$ and the circles $C_\rho :\ |z| = 1$, $C'_\rho :\ |1 - z| = 1$ with $0 < \rho \ll 1$. Here ρ is chosen small enough such that now poles of $R(z)$ are in interior or on of any of these two circles. It implies that all poles of $R(z)$ are exterior to to the dumbbell contour $\Gamma_\rho := L_1 \cup C_\rho \cup L_2 \cup C'_\rho$. Then the residue theorem implies that

$$I_\rho := \int\limits_{\Gamma_\rho} f(z)dz = 2\pi i \left[\sum_{k=1}^{n} Res_{z=z_k} f(z) + Res_{z=\infty} f(z) \right], \tag{6}$$

where $z_1, \ldots, z_n$ are the residues of $f(z)$ for $z \in \mathbb{C}$

The definition of Γ_ρ and (5) also imply that

$$\begin{aligned} I_\rho = \int\limits_{\Gamma_\rho} f(z)dz &= \int\limits_{L_1} f(z)dz + \int\limits_{C_\rho} f(z)dz + \int\limits_{L_2} f(z)dz + \int\limits_{C'_\rho} f(z)dz \\ &= \int\limits_{1-\rho-i0}^{\rho-i0} f(z)dz + \int\limits_{C_\rho} f(z)dz + \int\limits_{\rho+i0}^{1-\rho+i0} f(z)dz + \int\limits_{C'_\rho} f(z)dz \\ &= e^{i2\pi\alpha} \int\limits_{1-\rho+i0}^{\rho+i0} f(z)dz + \int\limits_{C_\rho} f(z)dz + \int\limits_{\rho+i0}^{1-\rho+i0} f(z)dz + \int\limits_{C'_\rho} f(z)dz. \end{aligned} \tag{7}$$

We prove that $\lim\limits_{\rho\to 0} \int\limits_{C_\rho} f(z)dz = 0$ as follows:

$$\left| \int\limits_{C_\rho} f(z)dz = 0 \right| \le \int\limits_{|C_\rho|} |f(z)||dz| \le \frac{M_1 \rho^\alpha}{(1-\rho)^\alpha} \int\limits_{|C_\rho|} |dz| = 2\pi \frac{M_1 \rho^\alpha}{(1-\rho)^{\alpha+1}} \to 0 \text{ as } \rho \to 0^+$$

because $-1 < \alpha < 1$. Here $M_1 = \max\limits_{C_\rho} |R(z)|$ and $|C_\rho|$ means that the integral is taken is the positive direction. In a similar way we prove that $\lim\limits_{\rho\to 0} \int\limits_{C'_\rho} f(z)dz = 0$.

Thus taking the limit $\rho \to 0^+$ in (7) and using (6) we obtain that

$$I = \frac{1}{1 - e^{i2\pi\alpha}} 2\pi i \left[\sum_{k=1}^{n} Res_{z=z_k} f(z) + Res_{z=\infty} f(z) \right]. \tag{8}$$

To find $Res_{z=\infty} f(z)$ we consider the Laurent series of $f(z)$ at $z = \infty$ by first finding Laurent series for $R(z)$ and $q(z) := \left(\frac{z}{1-z}\right)^\alpha$. For $R(z)$ we use (2) to obtain the Laurent series

$$R(z) = c_0 + \frac{c_{-1}}{z} + \frac{c_{-2}}{z^2} + \ldots, \; |z| > R_0, \tag{9}$$

where $R_0 > 0$ is chosen to be large nought such that all finite poles of $R(z)$ are located in $|z| < R_0$.

For $q(z)$ we obtain that

$$q(z) = \left(\frac{z}{1-z}\right)^\alpha = \left(-\frac{1}{1-\frac{1}{z}}\right)^\alpha = e^{i\alpha\pi} \left(\frac{1}{1-\frac{1}{z}}\right)^\alpha = e^{i\alpha\pi} \left[1 + \frac{\alpha}{z} + \ldots\right], \; |z| > 1, \tag{10}$$

where we used the Taylor series for $w := \frac{1}{z}$ and we moved from $z = x + i0, \; 0 < x < 1$ to $z = x \gg 1$ by moving around the branch point $z = 1$ in the negative direction on the argument $-\pi$ around the branch point $(1-z)^{-alpha}$ thus accumulating an addition to the argument of $q(z)$ as $(-\pi)(-\alpha) = \pi\alpha$ thus giving the factor $e^{i\alpha\pi}$.

Combining (9) and (10) we obtain the Laurent series for $f(z)$ as

$$f(z) = R(z)q(z) = e^{i\alpha\pi} \left[c_0 + \frac{\alpha c_0 + c_{-1}}{z} + \ldots\right], \; |z| > R_0, \tag{11}$$

which gives that

$$Res_{z=\infty} f(z) = -e^{i\alpha\pi}(\alpha c_0 + c_{-1}). \tag{12}$$

Together with (8) and (12) we thus evaluate the definite integral (1).