1. Find all roots of $(-i+\sqrt{3})^{1/2}$. Find expressions for real and imaginary parts of each root. Identify a principle value.

$$z = -i + \sqrt{3} = re^{i\theta}$$

$$r = \sqrt{1+3} = 2$$

$$\theta = -\text{Arctan}\frac{1}{\sqrt{3}} = -\frac{\pi}{6} = -\frac{\pi}{2} + \frac{\pi}{3}$$

$$z^{1/2} = \sqrt{2}\, e^{-\frac{i\pi}{2\cdot 6} + \frac{2\pi k}{2}}, \quad k = 0, 1$$

$$c_0 = \sqrt{2}\, e^{-i\frac{\pi}{12}} = \sqrt{2}\cos\frac{\pi}{12} - i\sqrt{2}\sin\frac{\pi}{12}$$

$$c_1 = -\sqrt{2}\, e^{-i\frac{\pi}{12}} = -\sqrt{2}\cos\frac{\pi}{2} + i\sqrt{2}\sin\frac{\pi}{12}$$

Also $e^{-\frac{i\pi}{2\cdot 6}} = e^{-\frac{i}{2}\left(\frac{\pi}{2} - \frac{\pi}{3}\right)} = e^{-i\frac{\pi}{4}} e^{i\frac{\pi}{6}}$

$$= \frac{(1-i)}{\sqrt{2}}\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right) = \frac{\sqrt{3}+1}{2\sqrt{2}} + i\frac{(-\sqrt{3}+1)}{2\sqrt{2}}$$

$\Rightarrow \text{Re}\, c_0 = \sqrt{2}\cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2}$ principle value

$\text{Im}\, c_0 = -\sqrt{2}\sin\frac{\pi}{12} = \frac{-\sqrt{3}+1}{2}$

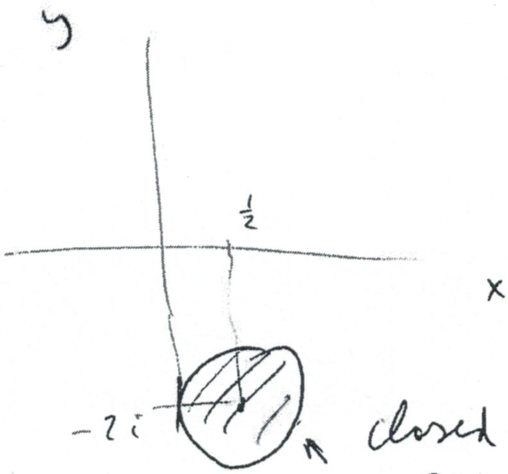
$\text{Re}\, c_1 = -\text{Re}\, c_0 = -\sqrt{2}\cos\frac{\pi}{2} = -\frac{\sqrt{3}+1}{2}$

$\text{Im}\, c_1 = -\text{Im}\, c_0 = \sqrt{2}\sin\frac{\pi}{12} = \frac{+\sqrt{3}-1}{2}$

2 (a) Sketch the following set and determine if it is open set, closed set or neither of them: $|2z - 1 + 4i| \leq 1$.
(b) Is this set bounded or unbounded?

$$\left| z - \frac{1}{2} + 2i \right| \leq \frac{1}{2}$$

Disk of radius $\frac{1}{2}$ centered at $z_0 = \frac{1}{2} - 2i$



closed set.
Set is bounded.

3. Determine if limit of $f(z)$ exists for $z \to 0$, where

$$f(z) = \left(\frac{\bar{z}}{z}\right)^3.$$

$z = x + i0$, $x \neq 0$

$$\left(\frac{x - i0}{x + i0}\right)^3 = 1$$

$z = 0 + iy$, $y \neq 0$

$$\left(\frac{0 - iy}{0 + iy}\right)^3 = -1$$

$\Rightarrow \lim_{z \to 0} \left(\frac{\bar{z}}{z}\right)^3$ does not exist.

(4)

4. Show by induction that n-th derivative of $f(z) = 1/z^2$ is given by
$f(z)^{(n)} = (-1)^n(n+1)!z^{-n-2}, n = 1, 2, \ldots, \; (z \neq 0)$.
Assume that expression for derivative of z^{-k} is known for any positive integer k.

By induction

① For $n = 1$

$$\left(\frac{1}{z^2}\right)' = -\frac{2}{z^3} = (-1)^n (n+1)!\, z^{-n-2}\Big|_{n=1}$$

② Assume that for $n = m$:

$$\left(\frac{1}{z^2}\right)^{(m)} = (-1)^m (m+1)!\, z^{-m-2}$$

$$\Rightarrow \left(\frac{1}{z^2}\right)^{(m+1)} = \left(\left(\frac{1}{z^2}\right)^{(m)}\right)' = \left((-1)^m (m+1)!\, z^{-m-2}\right)'$$

$$= (-1)^m (m+1)!(-m-2)\, z^{-m-3} = (-1)^{m+1} (m+2)!\, z^{-(m+1)-2}$$

$\blacksquare$

5. Find if $u(x,y) = 4xy - 2x$ is harmonic in some domain and if it is harmonic find a harmonic conjugate $v(x,y)$.

$u_{xx} + u_{yy} = 0 + 0 = 0 \Rightarrow$ harmonic in $\mathbb{C}$.

$v_x = -u_y = -4x$

$\Rightarrow v = -\int u_y \, dx = -2x^2 + \phi(y)$

$v_y = u_x$

$\Rightarrow \phi'(y) = 4y - 2$

$\Rightarrow \phi = 2y^2 + c - 2y$

$v = -2x^2 + 2y^2 + c - 2y$

6. Find all values of $\sinh^{-1}(4i)$.

$$W = \sinh^{-1}(z)$$

$$\Rightarrow z = \sinh W = \frac{e^{W} - e^{-W}}{2}$$

$$e^{2W} - 2z e^{W} - 1 = 0$$

$$e^{W} = z + (z^2+1)^{1/2}$$

$$W = \log\left(z + (z^2+1)^{1/2}\right)$$

$$\Rightarrow \sinh^{-1}(4i) = \log\left(4i + (-16+1)^{1/2}\right)$$

$$= \log\left(4i + (-15)^{1/2}\right)$$

(A) $\log(4i + i\sqrt{15}) = \ln(4+\sqrt{15}) + \frac{i\pi}{2} + 2\pi n i$, $n \in \mathbb{N}$

(B) $\log(4i - i\sqrt{15}) = \ln(4-\sqrt{15}) + \frac{i\pi}{2} + 2\pi n i$, $n \in \mathbb{N}$.

But $\ln(4-\sqrt{15}) = \ln \frac{16-15}{4+\sqrt{15}} = -\ln(4+\sqrt{15})$

$$\Rightarrow \sinh^{-1}(4i) = \pm \ln(4+\sqrt{15}) + i\left(\frac{\pi}{2} + 2\pi n\right),$$

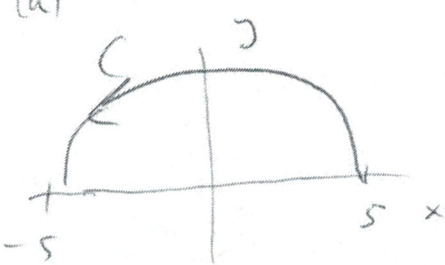
$n = 0, \pm 1, \pm 2, \ldots$

7. Assume that C is the positively oriented semicircle $z = 5e^{i\theta}$ $(0 \le \theta \le \pi)$ and $F(z)$ is the principle branch of $f(z) = z^{3-i}$.
(a) Use parametric representation for C to find

$$\int_C F(z)dz.$$

(b) What is a branch point for $f(z)$?

(a)



$$f(z) = z^{3-i} = e^{(3-i)\log z}$$

$$\Rightarrow F(z) = e^{(3-i)\operatorname{Log} z}$$

For $z = 5e^{i\theta}$

$$\operatorname{Log} z = \ln 5 + i\theta, \quad -\pi < \theta < \pi$$

$$\Rightarrow \int_C F(z)\,dz = \int_C e^{(3-i)(\ln 5 + i\theta)}\, d(5e^{i\theta})$$

$$= \int_0^{\pi} 5i\, e^{(3-i)\ln 5}\, e^{3i\theta + \theta + i\theta}\, d\theta$$

$$= 5i\, e^{(3-i)\ln 5} \frac{e^{(4i+1)\theta}}{4i+1}\Bigg|_{\theta=0}^{\theta=\pi} = \frac{5i}{4i+1} e^{(3-i)\ln 5}\left[e^{\pi}-1\right]$$

$$= \frac{(4i+1)5i}{17} e^{(3-i)\ln 5}\left[e^{\pi}-1\right] = \frac{5}{17}(i+4)e^{(3-i)\ln 5}\left[e^{\pi}-1\right]$$

(b) ~~Ant~~ branch point is a common point of all branch cuts $\Rightarrow$ $z=0$ is a branch point.