math 361

1

HW 01 Solutions

p. 5

1 (a) $(\sqrt{2}-i)-i(1-\sqrt{2}i)$

$= \sqrt{2}-i-i+\sqrt{2}i^2 = -2i$

(b) $(2,-3)(-2,1) = (-4+3, 6+2)$

$= (-1, 8)$

(c) $(3,1)(3,-1)(\frac{1}{5}, \frac{1}{10})$

$= (9+1, 3-3)(\frac{1}{5}, \frac{1}{10}) = (10,0)(\frac{1}{5}, \frac{1}{10})$

$= (2, 1)$

2 (a) $Re(iz) = Re(ix-y) = -y = -Im(z)$

(b) $Im(iz) = Im(ix-y) = x = Re\,z$

3 $(1+z)^2 = (1+z) + (1+z)z = 1+2z+z^2$

4 $z^2 = (1\pm i)^2 = 1 \pm 2i - 1 = \pm 2i$

$z^2 - 2z + 2 = \pm 2i - 2(1\pm i) + 2 = \pm 2i - 2 \mp 2i$
$+ 2 = 0$

5

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$$

$$= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \quad (1)$$

$$z_2 z_1 = (x_2 + iy_2)(x_1 + iy_1)$$

$$= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \quad (2)$$

$(1) = (2)$ ∎

6

(a) $$(z_1 + z_2) + z_3 = x_1 + x_2 + i(y_1 + y_2) + x_3 + iy_3 = x_1 + x_2 + x_3 + i(y_1 + y_2 + y_3)$$

$$= x_1 + (x_2 + x_3) + iy_1 + i(y_2 + y_3)$$

$$= z_1 + (z_2 + z_3) \quad ∎$$

(b) $$z_1(z_2 + z_3) = (x_1 + iy_1)(x_2 + iy_2 + x_3 + iy_3)$$

$$= x_1(x_2 + x_3) - y_1(y_2 + y_3) + i(y_1(x_2 + x_3) + x_1(y_2 + y_3)) = ((x_1 x_2 - y_1 y_2) + i(y_1 x_2 + x_1 y_2)) + ((x_1 x_3 - y_1 y_3 + i(y_1 x_3 + x_1 y_3)) = z_1 z_2 + z_1 z_3$$

∎

p. 8

1

(a) $\frac{1+2i}{3-4i} + \frac{2-i}{5i}$

$$= \frac{(1+2i)(3+4i)}{9+16} + \frac{(2-i)(-5i)}{25}$$

$$= \frac{3-8+i(6+4)}{25} + \frac{-10i-5}{25} = \frac{-5+10i-10i-5}{25}$$

$$= -\frac{10}{25} = -\frac{2}{5}$$

(b) $\frac{5i}{(1-i)(2-i)(3-i)} = \frac{5i}{(2-1+i(-2-1))(3-i)}$

$$= \frac{5i}{(1-3i)(3-i)} = \frac{5i}{\cancel{3}-10i} = \frac{5i(10i)}{-10i \cdot 10i} = -\frac{1}{2}$$

(c) $(1-i)^4 = ((1-i)^2)^2 = (1-2i-1)^2$

$$= (-2i)^2 = -4$$

2 $\quad z \neq 0$

$$\frac{1}{1/z} = \frac{1}{\bar{z}/|z|^2} = \frac{|z|^2}{\bar{z}} = \frac{|z|^2}{|z|^2} z = z.$$

3 $\quad (z_1 z_2)(z_3 z_4) = z_1 (z_2 z_3) z_4$

$$= z_1 (z_3 z_2) z_4 = (z_1 z_3)(z_2 z_4)$$

4 $\quad z_1 z_2 z_3 = (z_1 z_2) z_3 = 0$

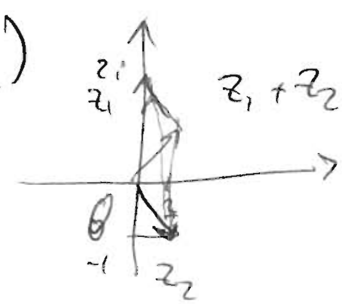
$\Rightarrow \quad z_1 z_2 = 0$ or $z_3 = 0$

$\Downarrow$

$z_1 = 0$ or $z_2 = 0$

$\Rightarrow \quad z_1 = 0$ or $z_2 = 0$ or $z_3 = 0$

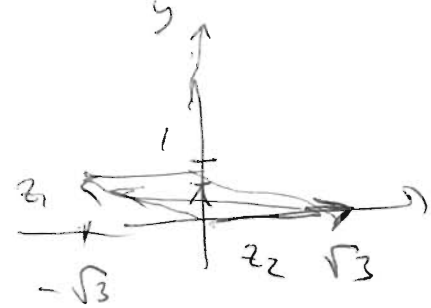
p 12 $\quad$ 1 $\quad$ a)



$z_1 + z_2 = \frac{2}{3} + i$

$z_1 - z_2 = -\frac{2}{3} + 3i$

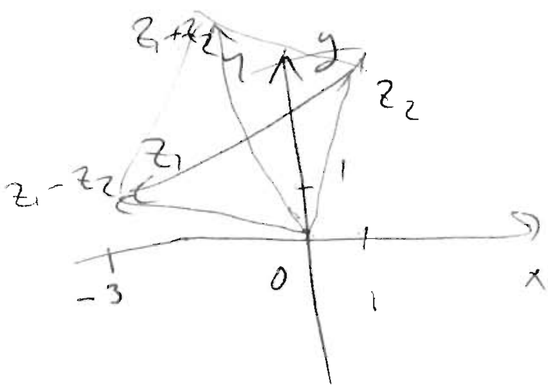
(b)



$z_1 + z_2 = (0, 1)$

$z_1 - z_2 = (-2\sqrt{3}, 1)$

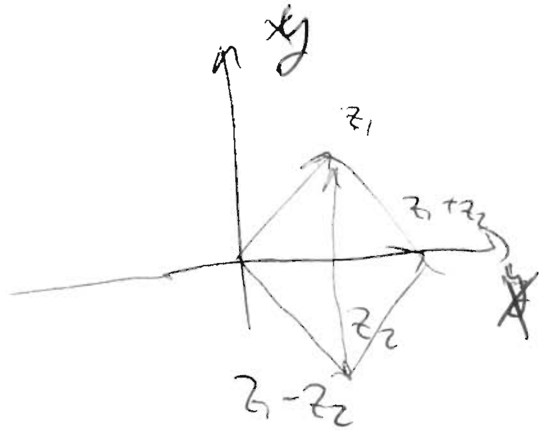
(c)



$z_1 + z_2 = (-2, 5)$

$z_1 - z_2 = (-4, -3)$

(d)



$z_1 + z_2 = 2x_1$

$z_1 - z_2 = 2iy_1$

2

$\operatorname{Re} z = x \le |x| = |\operatorname{Re} z| \le \sqrt{x^2+y^2} = |z|$

$\operatorname{Im} z = y \le |y| = |\operatorname{Im} z| \le \sqrt{x^2+y^2} = |z|$

3

$|z_3 + z_4| \ge ||z_3| - |z_4||$ $\qquad z_3 + z_4$

$\Rightarrow \dfrac{1}{|z_3+z_4|} \le \dfrac{1}{||z_3|-|z_4||}$

$$Re(z_1 + z_2) \leq |z_1 + z_2| \leq |z_1| + |z_2|$$

$$\Rightarrow \frac{Re(z_1+z_2)}{|z_3+z_4|} \leq \frac{|z_1|+|z_2|}{||z_3|-|z_4||}$$

4 $$(|x|-|y|)^2 \geq 0$$

But $$(|x|-|y|)^2 = x^2+y^2-2|x||y| \geq 0$$

$$\Rightarrow 2|z|^2 - |z|^2 - 2|x||y| \geq 0$$

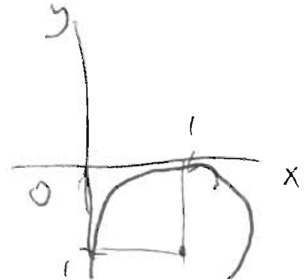
$$\Rightarrow 2|z|^2 \geq |z|^2 + 2|x||y|$$

$$= x^2+y^2+2|x||y| = (|x|+|y|)^2$$

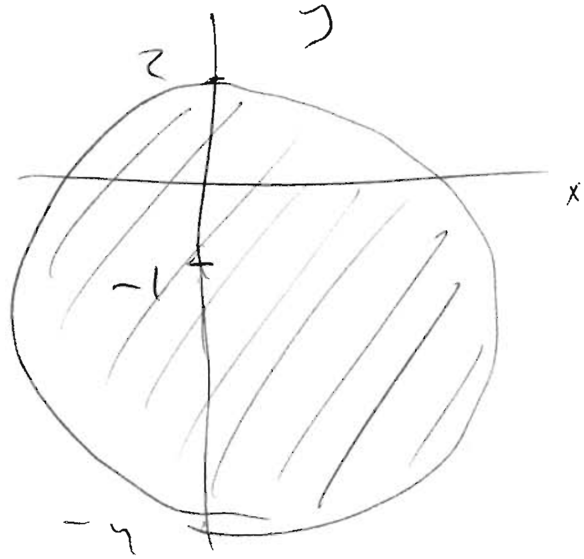
$$\Rightarrow \sqrt{2}\,|z| \geq ||x|+|y|| = |x|+|y| \quad \blacksquare$$

5 (a) $|z-1+i|=1$

$|z-(1-i)|=1$ – circle of radius 1 centered at $(1-i$



(b) $|z+i| \le 3$ – disc of radius 3 centered at $-i$

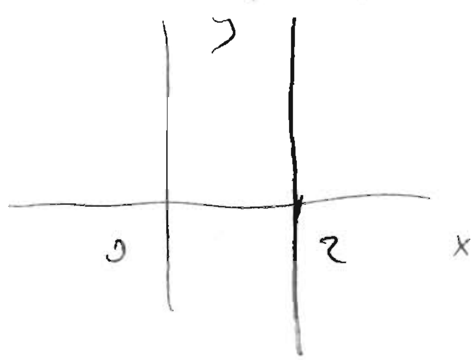


p.14 ① (a) $\overline{\bar{z}+3i} = \bar{\bar{z}} + \overline{3i}$

$= z - 3i$

(c) $\overline{(z+i)^2} = (2-i)^2 = 4 - 1 - 4i = 3 - 4i$

② (a) $Re(\bar{z} - i) = Re(x - iy - i) = x = 2$



$$|Re(z+\bar{z}+z^3)|$$

$$\leq |z+\bar{z}+z^3| \leq 2+|z|+|z|^3$$

{ since $|z|\leq 1 \Rightarrow |z|^3 \leq 1$

$$\leq 2+1+1=4 \quad \boxtimes$$

11 (a)

1 $n=2 \Rightarrow \overline{z_1+z_2} = \cancel{x}_1+x_2-i(y_1+y_2)$

$$=\bar{z}_1+\bar{z}_2$$

2 assume for $n=m$:

$$\overline{z_1+\dots+z_m} = \bar{z}_1+\dots+\bar{z}_n$$

$$\overline{z_1+z_2+\dots+z_{m+1}} = \overline{z_1+z_2+\dots+z_m}+\overline{z_{m+1}}$$

$$= \bar{z}_1+\dots+\overline{z_{m+1}} \quad \boxtimes$$

(15)

show $|z_1 + z_2| \le |z_1| + |z_2|$

$$|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1} + \overline{z_2})$$

$$= |z_1|^2 + |z_2|^2 + z_1\overline{z_2} + \overline{z_1} z_2$$

$$= |z_1|^2 + |z_2|^2 + 2\,Re(z_1\overline{z_2})$$

$$\le |z_1|^2 + |z_2|^2 + 2|z_1||z_2| = (|z_1| + |z_2|)^2$$

$$\Rightarrow |z_1 + z_2| \le |z_1| + |z_2| \quad \blacksquare$$

Homework 01

1. Prove by induction the following inequality

$$\left|\sum_{j=1}^{n} z_j\right| \le \sum_{j=1}^{n} |z_j|, \quad \text{where } z_j \in \mathbb{C} \text{ and } n \text{ is the positive integer.}$$

Pf $n=1$ trivial:

$$\left|\sum_{j=1}^{1} z_j\right| = |z_1|$$

Assume true for $n=m$

$$\left|\sum_{j=1}^{m+1} z_j\right| = \left|\sum_{j=1}^{m} z_j + z_{m+1}\right| \le$$

$$\le \left|\sum_{j=1}^{m} z_j\right| + |z_{m+1}| \le \sum_{j=1}^{m} |z_j| + |z_{m+1}|$$

by triag inequality

by

$$= \sum_{j=1}^{m+1} |z_j| \quad \blacksquare$$

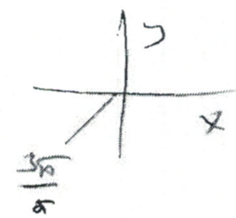
10

P.22 (1)

(a) $z = \dfrac{i}{-2-2i} = -\dfrac{1}{2}\,\dfrac{i}{1+i}$

$= -\dfrac{1}{2}\, \dfrac{i(1-i)}{2} = \dfrac{1}{4}(-1-i)$

$= \dfrac{1}{4} e^{i(-\frac{3}{4}\pi)}$ $\Rightarrow \operatorname{Arg} z = -\dfrac{3\pi}{4}$



(b) $z = (\sqrt{3}-i)^6$

$\sqrt{3} - i = \sqrt{4}\, e^{i\theta}$, $\theta = \arctan(-\frac{1}{\sqrt{3}})$

$= -\dfrac{\pi}{6}$

$\Rightarrow \operatorname{Arg}(\sqrt{3}-i) = -\dfrac{\pi}{6}$

$z = \sqrt{4}^{6} \cdot e^{-i\frac{\pi}{6}\cdot 6}$ $\Rightarrow \operatorname{Arg} z = -\pi + 2\pi = \pi.$

3

Show $e^{i\theta_1} \cdots e^{i\theta_n} = e^{i(\theta_1 + \cdots + \theta_n)}$

1 $n = 2$ $e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1 + \theta_2)}$

from section 8.

2 Assume for $m = n$ $e^{i\theta_1} \cdots e^{i\theta_m} = e^{i(\theta_1 + \cdots \theta_m)}$

$$e^{i\theta_1} \cdots e^{i\theta_m} e^{i\theta_{m+1}}$$

$$= e^{i(\theta_1 + \cdots + \theta_m)} e^{i\theta_{m+1}} = e^{i(\theta_1 + \cdots \theta_{m+1})}$$

$\square$

5 (a) $(1 - i\sqrt{3}) = 2e^{-i\frac{\pi}{3}}$, $\sqrt{3} + i = 2e^{i\frac{\pi}{6}}$,

$i = e^{i\frac{\pi}{2}}$

$$i(1 - i\sqrt{3})(\sqrt{3} + i) = 4e^{i\left(\frac{\pi}{2} - \frac{\pi}{3} + \frac{\pi}{6}\right)} = 4e^{i\frac{\pi}{3}}$$

$$= 4\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) + 2\left(1 + i\sqrt{3}\right)$$ 12

(c)

$$-1+i = \sqrt{2}\, e^{i\frac{3}{4}\pi}$$

$$(-1+i)^7 = (\sqrt{2})^7 \cdot e^{i\frac{21}{4}\pi}$$

$$= (\sqrt{2})^7 \cdot e^{-i\frac{3}{4}\pi} = (\sqrt{2})^7 \frac{-1-i}{\sqrt{2}} = \underline{-8(1+i)}.$$

(8) Assume that $|z_1| = |z_2|$

$$z_1 = r_1 e^{i\theta_1}$$
$$z_2 = r_2 e^{i\theta_2}$$
$$|z_1| = |z_2| \Rightarrow r_1 = r_2$$

choose $c_1 = \sqrt{r_1}\, e^{i\frac{\theta_1+\theta_2}{2}}$

$$c_2 = \sqrt{r_1}\, e^{i\frac{\theta_1-\theta_2}{2}}$$

$$\Rightarrow c_1 c_2 = r_1 e^{i\theta_1} = z_1$$

$c_1 \bar{c}_2 = r_1 e^{i\theta_2} = z_2$, i.e. such c_1, c_2

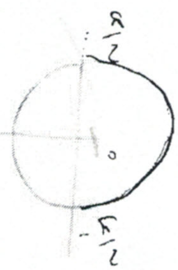
Assume

$z_1 = c_1 c_2$

$z_2 = c_1 \overline{c_2}$

$\Rightarrow |z_1| = |c_1||c_2| = |z_2|$ ∎

BC p24 ⑥ $\operatorname{Re} z_1 > 0$, $\operatorname{Re} z_2 > 0 \Rightarrow$



$$\left.\begin{array}{l} \Rightarrow -\frac{\pi}{2} < \operatorname{Arg} z_1 < \frac{\pi}{2} \\ -\frac{\pi}{2} < \operatorname{Arg} z_2 < \frac{\pi}{2} \end{array}\right\} \Rightarrow \quad \operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg} z_2 + 2\pi n,$$

where n such that

$$-\pi < \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) + 2\pi n < \pi$$

But $-\pi < \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) < \pi \Rightarrow n = 0$

⑪

(a) Binominal formula:

$$(z_1 + z_2)^n = \sum_{k=0}^{n} \binom{n}{k} z_1^k z_2^{n-k}, \quad n = 1, 2, \ldots$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad k = 0, 1, \ldots$$

(b) Moivre formula:

$$(\cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta, \quad n = 0, \pm 1, \ldots$$

From (a) and (b) for $z_2 = \cos\theta$, $z_1 = i \sin\theta \Rightarrow$

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

$$= \sum_{k=0}^{n} \binom{n}{k} \cos^{n-k}\theta\, (i\sin\theta)^k. \quad , n = 0, 1, 2, \dots$$

Pf

$$\Rightarrow \cos n\theta = Re\left(\cos n\theta + i\sin n\theta\right)$$

$$= \sum_{k=0}^{n} \binom{n}{k} \cos^{n-k}\theta\, (\sin\theta)^k\, Re\left(i^k\right)$$

If k is odd $\Rightarrow Re(i^k) = 0$

Pf omitting all odd k

$$\Rightarrow \cos n\theta = \sum_{k=0}^{m} \binom{n}{2k} \cos^{n-2k}\theta\, (\sin\theta)^{2k} (i)^{2k},$$

where $m = \begin{cases} n/2 & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$

But $(i)^{2k} = (-1)^k$

$$\Rightarrow \cos n\theta = \sum_{k=0}^{m} \binom{n}{2k} \cos^{n-2k}\theta\, (\sin\theta)^{2k} (-1)^k.$$

(b) $x = \cos\theta$

$$\sin^2\theta = 1-\cos^2\theta = 1-x^2$$

$$\Rightarrow (\sin\theta)^{2k} = (\sin^2\theta)^k = (1-x^2)^k$$

$$\Rightarrow \cos n\theta = \sum_{k=0}^{m} \binom{n}{2k} (-1)^k x^{n-2k} (1-x^2)^k$$

p 29 (1)

(a) $(2i)^{1/2}$

$$2i = 2e^{i\frac{\pi}{2}+2k\pi i}, \quad k=0,1$$

$$\Rightarrow (2i)^{1/2} = \sqrt{2}\, e^{i\frac{\pi}{4}+\pi i k} = \pm\sqrt{2}\, e^{i\frac{\pi}{4}}$$

$$= \pm\sqrt{2}\,\frac{1+i}{\sqrt{2}} = \pm(1+i)$$

(b) $1 - \sqrt{3}\, i = \sqrt{1+3}\, e^{-i\frac{\pi}{3}+2\pi i k}, \quad k=0,1$

$$\Rightarrow (1-\sqrt{3}\, i)^{1/2} = \sqrt{2}\, e^{-i\frac{\pi}{6}+\pi i k} = \pm\sqrt{2}\, e^{-i\frac{\pi}{6}}$$

$$= \pm\sqrt{2}\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right) = \pm\left(\frac{\sqrt{3}}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)$$

(2) (a) $(-16)^{1/4} = \left(16 \cdot e^{i\pi + 2\pi i k}\right)^{1/4}$

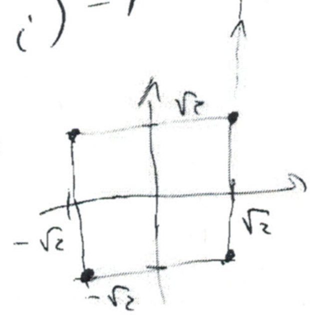
$= 2 \cdot e^{i\frac{\pi}{4} + \frac{\pi i k}{2}}, \quad k = 0, 1, 2, 3$

$k = 0: \quad 2\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}(1+i)$ - principle root

$k = 1: \quad 2\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}(-1+i)$

$k = 2: \quad 2\left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) = \sqrt{2}(-1-i)$

$k = 3: \quad 2\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \sqrt{2}(1-i)$

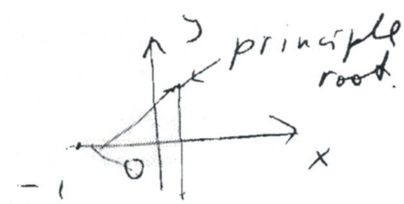


(3) (a) $(-1)^{1/3} = \left(1 \cdot e^{i\pi}\right)^{1/3} = 1 \cdot e^{i\frac{\pi}{3} + \frac{2\pi}{3}k}, \quad (k = 0, 1, 2)$

$k = 0: \quad \frac{1}{2} + i\frac{\sqrt{3}}{2}$

$k = 1: \quad e^{i\left(\frac{\pi}{3} + \frac{2\pi}{3}\right)} = -1$

$k = 2: \quad e^{i\left(\frac{\pi}{3} + \frac{4\pi}{3}\right)} = e^{i\frac{5}{3}\pi} = \frac{1}{2} - i\frac{\sqrt{3}}{2}$



(4) $\omega_3 = e^{i\frac{2\pi}{3}} = \frac{-1+\sqrt{3}i}{2}$

$z_0 = -4\sqrt{2} + 4\sqrt{2}\,i = 8\left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = 8e^{i\frac{3}{4}\pi}$

$z_0^{1/3} = 2e^{i\frac{\pi}{4} + \frac{2\pi i k}{3}}, \quad k = 0, 1, 2$

$\Rightarrow c_0 = 2e^{i\frac{\pi}{4}} = \sqrt{2}(1+i)$

$$c_1 = \omega_3 \cdot c_0 = e^{i\frac{2\pi}{3}} \cdot 2e^{i\frac{\pi}{4}} = 2(1+i)\left(\frac{-1+\sqrt{3}\,i}{2}\right)$$

$$= \frac{1}{\sqrt{2}}\left(-1-\sqrt{3}-i+\sqrt{3}\,i\right)$$

$$= \frac{-(\sqrt{3}+1)+(\sqrt{3}-1)i}{\sqrt{2}}$$

$$c_2 = \omega_3^2 c_0 = \omega_3 c_1 = \frac{-1+\sqrt{3}\,i}{2} \cdot \frac{-(\sqrt{3}+1)+(\sqrt{3}-1)i}{\sqrt{2}}$$

$$= \frac{\sqrt{3}+1-3+\sqrt{3}+i\left(-3-\sqrt{3}-\sqrt{3}+1\right)}{2\sqrt{2}}$$

$$= \frac{2\sqrt{3}-2+i\left(-2-2\sqrt{3}\right)}{2\sqrt{2}} = \frac{\sqrt{3}-1+i\left(-1-\sqrt{3}\right)}{\sqrt{2}}$$

5 (a) $z = a+i$

$|z| = \sqrt{a^2+1} = A$

$\alpha = \operatorname{Arg}(A+i)$

$z^{1/2} = \sqrt{A}\, e^{i\left(\frac{\alpha}{2}+\pi k\right)}$, $(k=0,1)$

$= \pm\sqrt{A}\, e^{i\frac{\alpha}{2}}$

(6)

(b) $\cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2}$ $\quad \sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2}$

$\Rightarrow \cos\frac{\alpha}{2} = \sqrt{\frac{1+\cos\alpha}{2}}$ $\quad \sin\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{2}}$

$$\pm\sqrt{A}\, e^{i\frac{\alpha}{2}} = \pm\sqrt{A}\left(\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\right)$$

$$a - i = A\cos\alpha + i A\sin\alpha$$

$\Rightarrow \cos\alpha = \frac{a}{A}$ $\quad \sin\alpha = -\frac{1}{A}$

$$\Rightarrow \pm\sqrt{A}\, e^{i\frac{\alpha}{2}} = \pm\sqrt{A}\left(\frac{\sqrt{1+\cos\alpha}}{\sqrt{2}} + i\frac{\sqrt{1-\cos\alpha}}{\sqrt{2}}\right)$$

$$= \pm\frac{\sqrt{A}}{\sqrt{2}}\left(\sqrt{1+\frac{a}{A}} + i\sqrt{1-\frac{a}{A}}\right) = \frac{\pm 1}{\sqrt{2}}\left(\sqrt{A+a} + i\sqrt{A-a}\right)$$

(7) (optional)

$$1 + c + \dots + c^{n-1} = \frac{1-c^n}{1-c} = 0 \text{ because } c^n = 1$$

(8) (a) $az^2 + bz + c = 0$

$$z^2 + \frac{2b}{2a}z + \frac{c}{a} = 0$$

$$z^2 + 2\frac{b}{2a}z + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$

$$\left(z+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

$$\Rightarrow \quad z+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-b+(b^2-4ac)^{1/2}}{2a}$$

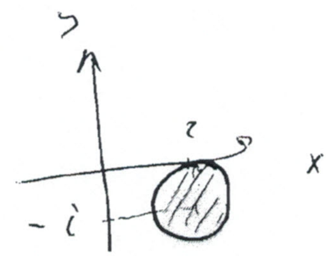
(b)

$$z^2+2z+(1-i)=0$$

$$z=\frac{-2+(4-4(1-i))^{1/2}}{}=-1+(1-(1-i))^{1/2}$$

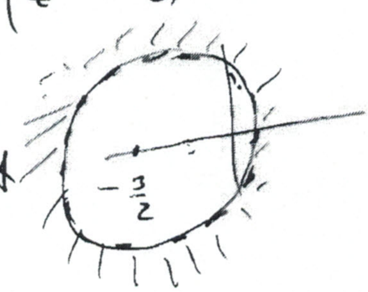
$$=-1+i^{1/2}=-1+e^{\frac{i\pi}{4}+\pi k}=-1\pm\frac{(1+i)}{\sqrt{2}}$$

p. 33 ① (a) $|z-2+i|\le 1$



closed set $\Rightarrow$ not a domain

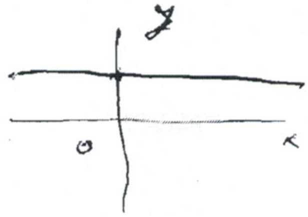
(b) $|2z+3|>4 \Rightarrow \left|z-\left(-\frac{3}{2}\right)\right|>2$



open, connected, non-empty set

$\Rightarrow$ domain

(d) $\text{Im}\, z = 1$



set is not open or closed if we include ∞ (contains boundary but not except ∞)

If neglect $\infty \Rightarrow$ closed set

$\Rightarrow$ not a domain in both cases.

(f) $|z-4| \geq |z|$

$|z-4|^2 \geq |z|^2$

$(x-4)^2 + y^2 \geq x^2 + y^2$

$\Rightarrow -8x + 16 \geq 0$

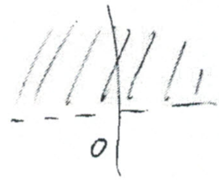
$x \leq 2$

$\Rightarrow$ closed set $\Rightarrow$ not a domain

② (a) closed

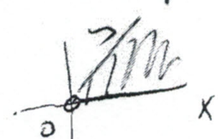
(b) open

(c) $\text{Im}\, z > 1$ $\Rightarrow$ open set



(d) If include $\infty \Rightarrow$ neither open or closed. If disregard $\infty \Rightarrow$ closed.

(e) $0 \leq \arg z \leq \frac{\pi}{4}, z \neq 0$



neither open or closed.

(f) closed set

3 (a) bounded

(b) unbounded (include arbitrary large $|z|$)

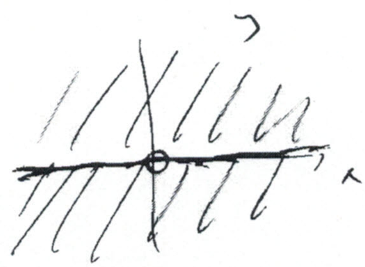
(c) unbounded

(d) unbounded

(e) unbounded

(f) unbounded.

4 (a) $-\pi < \arg z < \pi \ , \ z \neq 0$



$\Rightarrow$ closure

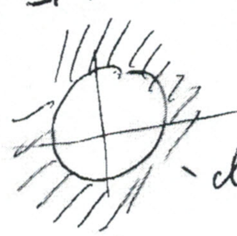
entire z-plane.

(c) $\mathrm{Re}\left(\frac{1}{z}\right) \le \frac{1}{2}$

$$\mathrm{Re}\,\frac{1}{x+iy} = \mathrm{Re}\,\frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} \le \frac{1}{2}$$

$$x^2+y^2 \geq 2x$$

$$(x-1)^2+y^2-1 \geq 0 \Rightarrow |z-1| \geq 1$$



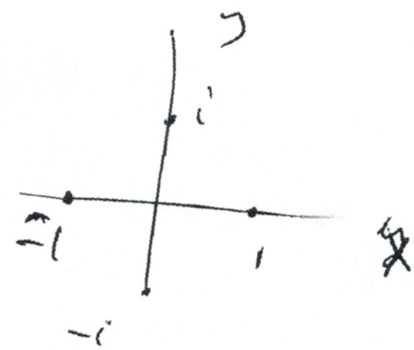
closure

⑦ (a)

$$z_n = i^n \quad (n = 1, 2, \ldots)$$

$$\left.\begin{aligned} z_{4m} &= 1 \\ z_{4m+1} &= i \\ z_{4m+2} &= -1 \\ z_{4m+3} &= -i \end{aligned}\right\}, \quad m = 1, 2, \ldots$$

$\Rightarrow$ S

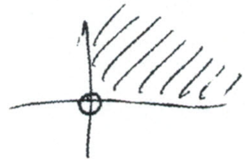


$\Rightarrow$ For any ~~neighb~~ deleted neighborhood of $i, -i, 1, -1$ we have no points of S $\Rightarrow$ not accumulation points.

For all other points z we choose $\varepsilon > 0$ small enough so that $0 < |z - z_0| < \varepsilon$ does not contain $i, -1, 1, -1$ $\Rightarrow$ not accumulation points.

$\Rightarrow$ no accumulation points.

(c) $0 \leq \arg z < \frac{\pi}{2}, \; z \neq 0$



$\Rightarrow$ All points of set are accumulation points.

8 $\forall$ Boundary point is the accumulation point $\Rightarrow$ if all accumulation points are included $\Rightarrow$ all boundary points are included $\Rightarrow$ set is closed

9 (optional)

Domain is open nonempty connected set

$\Rightarrow$ take $z_0 \in S \Rightarrow$ ~~th~~ z_0 is not a boundary point $\Rightarrow$ z_0 is interior point

$\Rightarrow \exists$ neighborhood containing only points of S

$\Rightarrow$ all smaller neighborhoods of that point contains points from $S \Rightarrow$ accumulation point.