math 561 Complex analysis. (1)

Hw 02

p.37. (1) (a) $f(z) = \frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}$

$\Rightarrow$ domain of definition is all complex plane $\mathbb{C}$ except $z = \pm i$

(d) $f(z) = \frac{1}{1 - z\bar{z}} \Rightarrow$ need $|z|^2 \neq 1$

$\Rightarrow z \neq e^{i\theta}, \ -\pi < \theta \leq \pi$

$\Rightarrow$ domain of definition is $\mathbb{C}$ except ring $|z| = 1$

(2) $z^3 + z + 1 = (x+iy)^3 + x + iy + 1$

$= x^3 - iy^3 + 3x^2 iy - 3xy^2 + x + iy + 1$

$= (x^3 - 3xy^2 + x + 1) + i\,(3x^2 y - y^3 + y)$

(4) $f = z + \frac{1}{z} = r e^{i\theta} + \frac{1}{r} e^{-i\theta}$

$= (r + \frac{1}{r}) \cos\theta + i\,(r - \frac{1}{r}) \sin\theta$

p 44

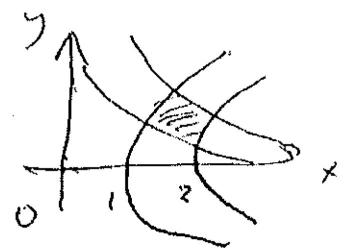
① $x^2-y^2+i2xy=1 \Rightarrow x^2-y^2=1$
$2xy=0$

$y=\pm\sqrt{x^2-1}$, $x\geq 1$

$x^2-y^2+2ixy=2 \Rightarrow y=\pm\sqrt{x^2-2}$

$x^2-y^2+2ixy=i \Rightarrow y=\frac{1}{2x}$

$x^2-y^2+2ixy=2i \Rightarrow y=\frac{1}{x}$

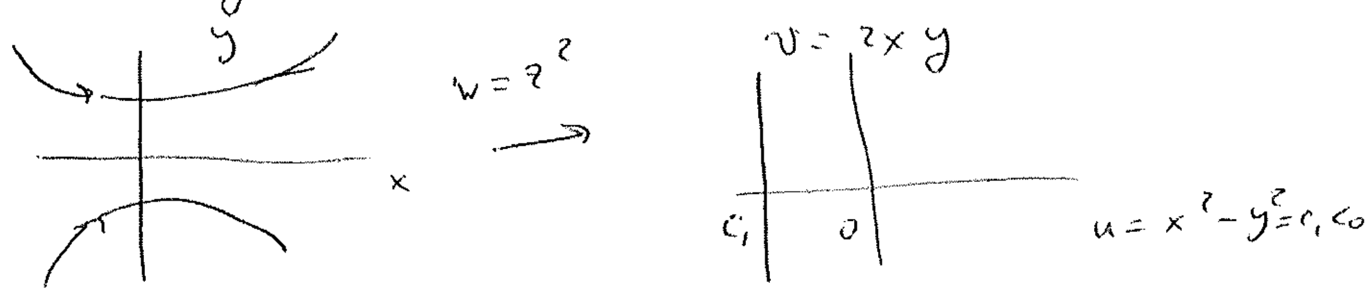


② $x^2-y^2=c_1$ $(c_1<0)$

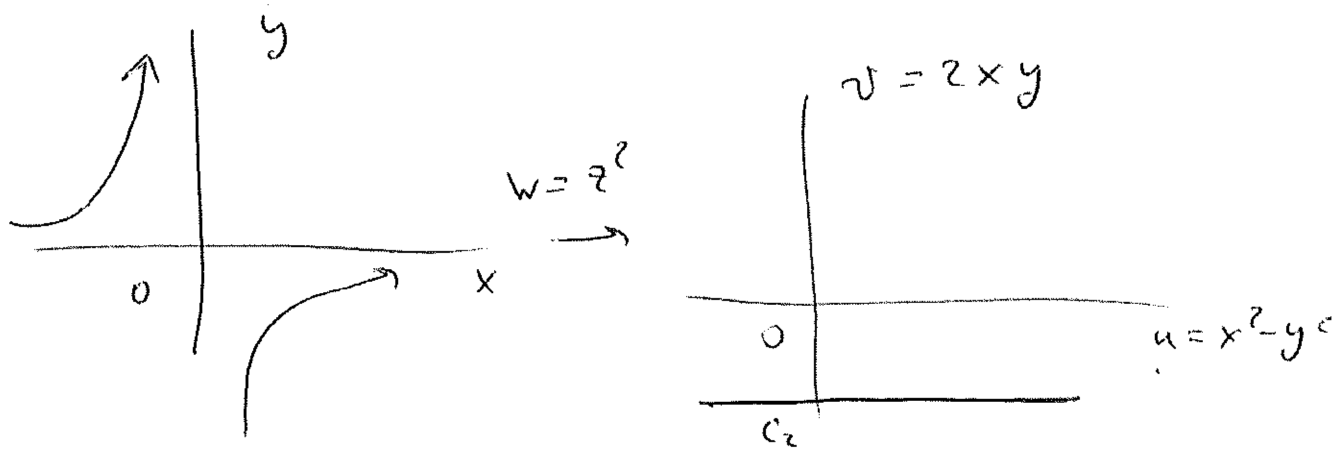
$w=z^2=x^2-y^2+2ixy$

$x^2-y^2=c_1<0$

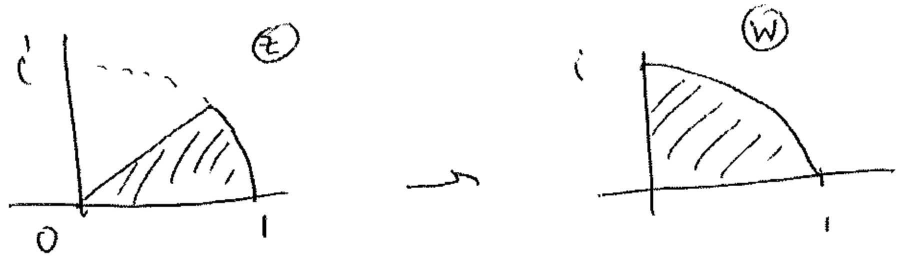
$\Rightarrow y=\pm\sqrt{x^2-c_1}=\pm\sqrt{x^2+|c_1|}$



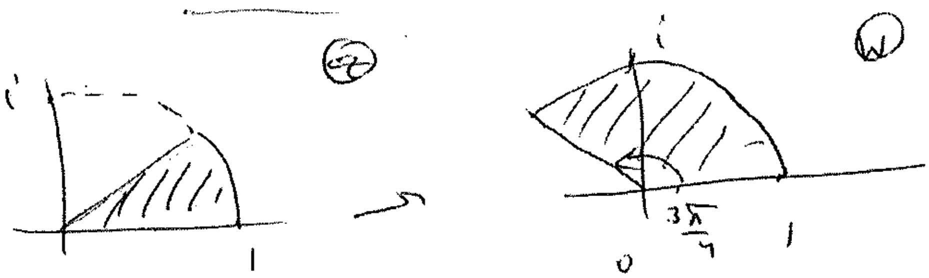
$2xy = c_2 < 0$



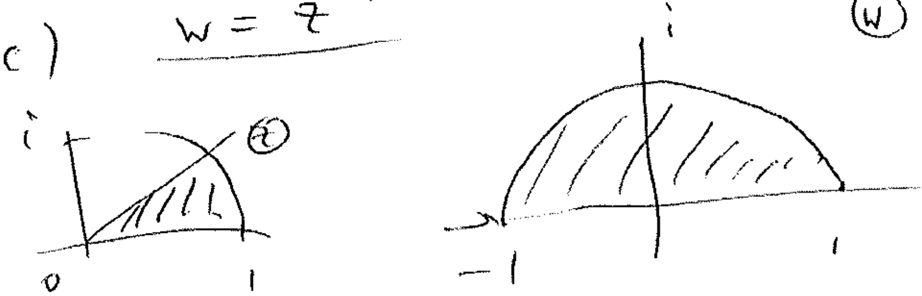
③ (a) $w = z^2$ $\qquad r \leq 1,\ 0 \leq \theta \leq \frac{\pi}{4}$



(b) $w = z^3$



(c) $w = z^4$



(5) Horizontal segments

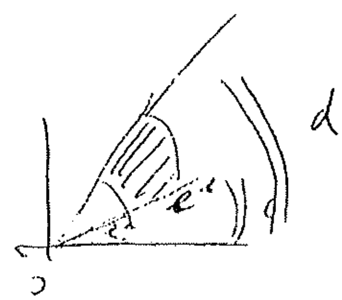
$z_1 = x + ic$

$z_2 = x + id$

$e^{z_1} = e^x e^{ic}$

$e^{z_2} = e^x e^{id}$

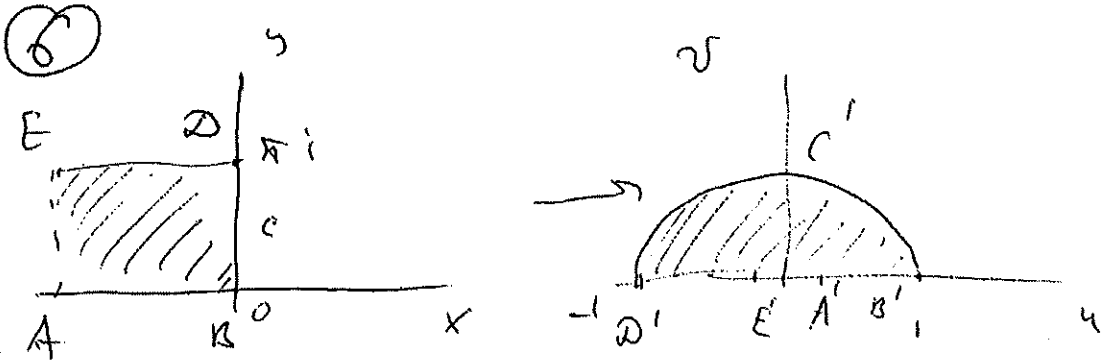
$\Rightarrow e^a \le e^x \le e^b$



$\arg e^{z_1} = c \quad \arg e^{z_2} = d$

$\Rightarrow c \le \arg z \le d$

(6)



$w = e^{z}$

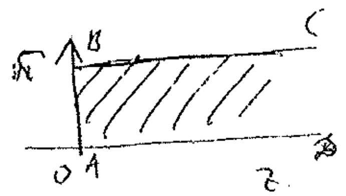
$B: z_0 = 0 \Rightarrow w_B = e^0 = 1$

$C' \Rightarrow$ segment$_{[0, i\pi]}$ BD : $z = e^{i\varphi} \Rightarrow$ circle C'

AB: $z = x \Rightarrow w = e^x \Rightarrow A'B'$ – real line between 1 at B' and $e^x \to 0$ at A' as $x \to -\infty$

ED: $z = x + i\pi \Rightarrow w = e^{x+i\pi} = -e^x$ – real line between -1 and E' with $e^x \to 0$ as $x \to -\infty$.

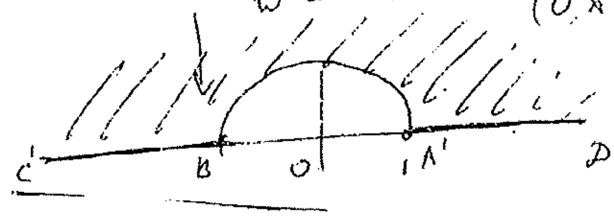
7



$x \geq 0, \; 0 \leq y \leq \pi$

$w = e^z$

$(0, \pi i)$ transforms into $1 \leq \rho \leq \infty$



p.55

① (a) Prove that $\lim_{z \to z_0} Re(z) = Re(z_0)$

Take $\delta = \varepsilon > 0$. Assume that $|z - z_0| < \delta$.

then $|Re(z) - Re(z_0)| \leq |z - z_0| < \varepsilon$ ▨

(b) Prove that $\lim_{z \to z_0} \bar{z} = \bar{z_0}$

Take $\delta = \varepsilon > 0$

Assume that $|z - z_0| < \delta$

then $|\bar{z} - \bar{z_0}| = |z - z_0| < \varepsilon$ ▨

② (a) Show that

$$\lim_{z \to z_0} (az + b) = az_0 + b$$

Take $\delta = \frac{\varepsilon}{|a|}$ for $|a| \neq 0$. Assume that $|z - z_0| < \delta$.

then $|az + b - az_0 - b| = |a|\,|z - z_0| < |a|\delta = \varepsilon$.

For $a = 0$ limit is trivial. ▨

(b) Show that $\lim_{z \to z_0} (z^2 + c) = z_0^2 + c$

Assume that $|z - z_0| < \delta$

then

$$|z^2+c-z_0^2-c| = |z^2-z_0^2| = |(z-z_0)(z+z_0)|$$

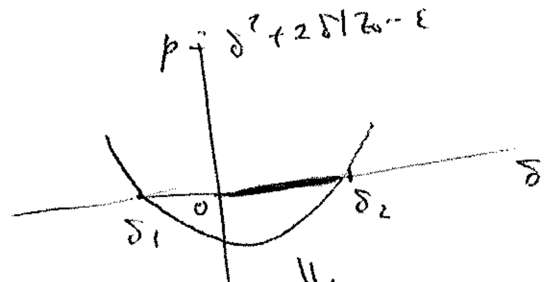
$$= |z-z_0|\,|z-z_0+2z_0| \leq |z-z_0|\left(|z-z_0|+2|z_0|\right)$$

$$< \delta(\delta+2|z_0|) < \varepsilon$$

we have to find δ such that this inequality is valid, i.e. $\delta^2+2\delta|z_0|-\varepsilon<0$

Solution of quadratic equation for δ: $p = \delta^2+2\delta|z_0|-\varepsilon = 0$

$$\delta_{1,2} = -|z_0| \pm \sqrt{|z_0|^2+\varepsilon}$$



$\Downarrow$ to ensure $p<0 \Rightarrow 0<\delta<\delta_2$

$$\delta_2 = -|z_0| + \sqrt{|z_0|^2+\varepsilon}$$

So we choose any $\delta>0$ s.t.

$$\delta < -|z_0| + \sqrt{|z_0|^2+\varepsilon}$$

$\blacksquare$

③ (b) $\lim_{z\to i} \frac{iz^3-1}{z+i} = \frac{\lim_{z\to i}(iz^3-1)}{\lim_{z\to i}(z+i)} = \frac{-i^2-1}{2i} = \frac{0}{2i} = 0.$

④ $\lim_{z\to z_0} z^n = z_0^n$

Pf by induction

1) For $n=1$ $\lim_{z\to z_0} z = z_0$

2) Assume $\lim_{z\to z_0} z^m = z_0^m$

$\Rightarrow \lim_{z\to z_0} z^{m+1} = \lim_{z\to z_0} z^m \lim_{z\to z_0} z = z_0^m \cdot z_0 = z_0^{m+1}$ ■.

⑤ Approaching along $z = x+i0$

$\Rightarrow \left(\frac{z}{\bar z}\right)^2 = \left(\frac{x}{\bar x}\right)^2 = 1$

Approaching along $z = x+ix$

$\Rightarrow \left(\frac{z}{\bar z}\right)^2 = \left(\frac{x+ix}{x-ix}\right)^2 = \left(\frac{e^{i\frac{\pi}{4}}}{e^{-i\frac{\pi}{4}}}\right)^2 = -1$

$\Rightarrow$ limit does not exist.

9. Show that

$$\lim_{z \to z_0} f(z) g(z) = 0 \quad \text{if} \quad \lim_{z \to z_0} f(z) = 0$$

and $\exists M > 0$ s.t. $|g(z)| \le M$ for $\forall z$.
provided $|z - z_0| < \delta_1$
for some $\delta_1 > 0$.

Pf. $\lim_{z \to z_0} f(z) = 0$ implies that

$\forall \frac{\varepsilon}{|M|} > 0$ $\exists \delta > 0$ such that $|z - z_0| < \delta$ implies that $|f(z)| < \frac{\varepsilon}{|M|}$.

Assume that $|z - z_0| < \delta$ and $\delta < \delta_1$.

then $|f(z) g(z)| = |f(z)||g(z)| < \frac{\varepsilon}{M} |g(z)|$

$$\le \frac{\varepsilon M}{M} = \varepsilon \quad \blacksquare$$

10

(a) $$\lim_{z \to \infty} \frac{4z^2}{(z-1)^2} = \left\{ \text{by thm of sect 17} \right\}$$

$$= \lim_{z \to 0} \frac{4}{z^2} \frac{1}{\left(\frac{1}{z}-1\right)^2} = \cancel{\lim_{z \to 0} \frac{\frac{4}{z^2}}{\frac{1}{z}-1}}$$

$$= \lim_{z \to 0} \frac{4}{(1-z)^2} = \frac{4}{1} = 4$$

(c) $$\lim_{z \to \infty} \frac{z^2+1}{z-1} = \lim_{z \to 0} \frac{\frac{1}{z^2}+1}{\frac{1}{z}-1} = \frac{\left(\lim_{z \to 0} \frac{1}{z}\right) \lim_{z \to 0} (1+z^2)}{\lim_{z \to 0} 1-z}$$

$$= \lim_{z \to 0} \frac{1}{z} = \infty$$

(11)

P.62

(1) (a) $f(z) = 3z^2 - 2z + 4$

$f'(z) = 3(z^2)' - 2(z') + (4)' = 6z - 2$

(c)

$f(z) = \dfrac{z-1}{2z+1}, \quad z \neq \dfrac{1}{2}$

$f'(z) = \dfrac{1(2z+1) - (z-1)2}{(2z+1)^2} = \dfrac{3}{(2z+1)^2}$

(d) $f(z) = \frac{(1+z^2)^4}{z^2}, \quad z \neq 0$

$f'(z) = \frac{2z4(1+z^2)^3 z^2 - (1+z^2)^4 2z}{z^4} = \frac{8(1+z^2)^3}{z}$

$- 2\frac{(1+z^2)^4}{z^3}$

(2)

a) $p(z) = a_0 + a_1 z + \ldots + a_n z^n, \quad a_n \neq 0$

Show by induction that

$p'(z) = a_1 + 2a_2 z + \ldots + n a_n z^{n-1}$

(1) $n = 1$

$\Rightarrow p'(z) = (a_0 + a_1 z)' = a_1$

(2) Assume for $n = m$ $\quad p'(z) = a_1 + 2a_2 z + \ldots + m a_m z^{m-1}$

Show for $n = m+1$:

$p'(z) = (a_0 + a_1 z + \ldots + a_m z^m + a_{m+1} z^{m+1})'$

$= (a_0 + \ldots + a_m z^m)' + (a_{m+1} z^{m+1})'$

$= a_1 + 2a_2 z + \ldots + m a_m z^{m-1} + (m+1) a_{m+1} z^m$

(b) Show by induction that
$a_0 = p(0)$, $a_1 = \frac{p'(0)}{1!}$, ... $a_n = \frac{p^{(n)}}{n!}$

First show by induction that

$$(z^n)^{(k)} = n(n-1)\dots(n-k+1)z^{n-k}, \quad k \le n$$

(1) $k=1$. $(z^n)^k = n z^{n-1}$

(2) Assume for $\underset{k \le n}{k=m}$: $(z^n)^{(k)} = n(n-1)\dots(n-k+1)z^{n-k}$

$$\Rightarrow (z^n)^{(k+1)} = n(n-1)\dots(n-k+1)(z^{n-k})'$$

$$= n(n-1)\dots(n-(k+1)-1)\, z^{n-k-1}$$

$$\Rightarrow (z^n)^{(k)}\Big|_{z=0} = 0 \text{ if } k < z.$$

$$(z^n)^{(n)} = n!$$

$$(z^n)^{(m)} = \left((z^n)^{(n)}\right)^{(m-n)} = 0 \quad \text{if } m > n.$$

$$\Rightarrow p^{(n)}(0) = n!\,a_n \Rightarrow a_n = \frac{p^{(n)}(0)}{n!} \;\forall n \ge 0$$

3 $w = \frac{1}{z}$, $z \ne 0$

$$w' = \lim_{\Delta z \to 0} \frac{\frac{1}{z+\Delta z} - \frac{1}{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{z-(z+\Delta z)}{z(z+\Delta z)\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{-\Delta z}{z(z+\Delta z)\Delta z} = \lim_{\Delta z \to 0} \frac{-1}{z(z+\Delta z)} = -\frac{1}{z^2}$$

(8) (a) $(z^n)' = nz^{n-1}$.

Show by induction

(1) for $n=1$: $(z)' = 1$ - true.

(2) Suppose that for $m = n$ $(z^m)' = mz^{m-1}$

then for $n = m+1$:

$$(z^{m+1})' = (z \cdot z^m)' = z' z^m + z(z^m)'$$

$$= z^m + m z \cdot z^{m-1} = (m+1) z^m$$

(8 (a)) $f(z) = \operatorname{Re} z$. Show that $f'(z)$ does not exist for any z.

If approaching z along Im axis:

$$\Delta z = i\Delta y$$

$$\lim_{\Delta y \to 0} \frac{\operatorname{Re}(z + i\Delta y) - \operatorname{Re} z}{i\Delta y} = \lim_{\Delta y \to 0} \frac{\operatorname{Re} z - \operatorname{Re} z}{i\Delta y} = 0$$

If approaching along Re axis:

$\Delta z = \Delta x$

$$\lim_{\Delta x \to 0} \frac{\operatorname{Re}(z + \Delta x) - \operatorname{Re} z}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

$\Rightarrow \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$ does not exist and f' is not differentiable $\forall z$.

(9)

$$f(z) = \begin{cases} \frac{\bar{z}^2}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

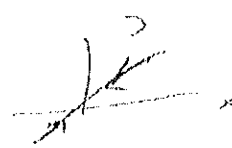
$$\Delta w = f(z) - f(0) = \frac{\bar{z}^2}{z}$$

(a) $\Delta z = \Delta x$ – along real axis

$$\lim_{\Delta x \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta x \to 0} \frac{\Delta x^2}{(\Delta x)^2} = 1$$

(b) $\Delta z = i\Delta y$ – along imaginary axis

$$\lim_{\Delta y \to 0} \frac{\Delta w}{i\Delta y} = \lim_{\Delta y \to 0} \frac{(-i\Delta y)^2}{(i\Delta y)^2} = 1$$



(c) Along $\Delta x = \Delta y$

limit $\Rightarrow \Delta z = \Delta x + i\Delta x$

$$\lim_{\Delta z \to 0} \frac{\bar{z}^2}{(\Delta z)(\Delta z)} = \lim_{\Delta x \to 0} \frac{(\Delta x - i\Delta x)^2}{(\Delta x + i\Delta x)^2}$$

$$= \lim_{\Delta x \to 0} \frac{(1-i)^2}{(1+i)^2} = \left(\frac{1-i}{1+i}\right)^2 = \frac{(1-i)^4}{2^2} = -1$$

$\Rightarrow f'(0)$ does not exist.