Math 501, HW03 Solutions

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1

(a) $f(z) = \bar{z} = x - iy$

$\Rightarrow u = x,\ v = -y$

$u_x = 1,\ v_y = -1 \Rightarrow \underline{u_x \neq v_y}$

$\Rightarrow f'(z)$ does not exist $\forall z$.

(b) $f(z) = z - \bar{z} = 2iy,\ u = 0,\ v = 2y$

$u_x = 0,\ v_y = 2 \Rightarrow u_x \neq v_y$

(c) $f(z) = 2x + ixy^2 \Rightarrow u = 2x,\ v = xy^2$

$u_x = 2,\ v_y = 2xy \Rightarrow u_x = v_y$ for $xy = 1$

$u_y = 0,\ v_x = y^2 \Rightarrow u_y = -v_x$ for $y = 0$

these conditions cannot be true simultaneously $\Rightarrow f'(z)$ does not exist $\forall z$.

(d) $f(z) = e^x e^{-iy} \Rightarrow u = e^x \cos y,\ v = -e^x \sin y$

$u_x = e^x \cos y,\ v_y = -e^x \cos y$

$\Rightarrow u_x = v_y$ i $e^x \cos y = -e^x \cos y$ only if $\cos y = 0$

or $y = \frac{\pi}{2} + \pi n,\ (1)$

$n = 0, \pm 1 \dots$

$u_y = -e^x \sin y \qquad v_x = -e^x \sin y$

$\Rightarrow u_y = -v_x \Rightarrow -e^x \sin y = -(-e^x \sin y)$

$\Rightarrow \sin y = 0 \Rightarrow y = \pi n, \ n = 0, \pm 1, \dots$ (2)

Conditions (1) and (2) cannot be satisfied simultaneously $\Rightarrow f'(z)$ does not exist $\forall z$

(2) (a) $f(z) = iz + 2 = 2 + ix - y$

$u = 2 - y$

$v = x$

conditions of ~~thm 22~~ thm of sec 22

(1) $u_x = 0, \ u_y = -1$
$v_x = 1, \ v_y = 0$
– exist for $\forall x, y$ and continuous

(2) $u_x = v_y = 0$
$u_y = -v_x = 1$
Cauchy-Riemann equations are satisfied

$\Rightarrow$ by thm of Sect 22

$f'(z)$ exists $\forall z$

and $f'(z) = u_x + i v_x = i$

For $g = f'(z) = i$ ③

Conditions of thm of sec 22

(1) $u = 0, v = 1 \Rightarrow u_x = u_y = v_x = v_y = 0$

— exist and continuous $\forall z$.

(2) $u_x = v_y = 0$
$u_y = -v_x = 0$ — Cauchy - Riemann equations are satisfied

$\Rightarrow f''(z)$ exists $\forall z$.

and $f''(z) = u_x + i v_x = 0$

~~2a~~ ~~$f(z) = x^2 + iy^2$~~

2d $f(z) = \cos x \cosh y - i \sin x \sinh y$

$u = \cos x \cosh y$
$v = -\sin x \sinh y$

(1)
$u_x = -\sin x \cosh y$
$u_y = \cos x \sinh y$
$v_x = -\cos x \sinh y$
$v_y = -\sin x \cosh y$
exist and continuous $\forall z$.

(2)
$u_x = v_y = -\sin x \cosh y$
$u_y = -v_x = \cos x \sinh y$
Cauchy-Riemann eqn. are satisfied

$\Rightarrow$ by th of Sec 22 $f'(z)$ exists $\forall z$

and $f'(z) = u_x + i v_x$

$= -\sin x \cosh y - i \cos x \sinh y$

Now for $g = f'(z) = -\sin x \cosh y - i \cos x \sinh y$

$u = -\sin x \cosh y$

$v = -\cos x \sinh y$

(1)
$$\left.\begin{aligned} u_x &= -\cos x \cosh y \\ u_y &= -\sin x \sinh y \\ v_x &= \sin x \sinh y \\ v_y &= -\cos x \cosh y \end{aligned}\right\} \text{exists and continuous } \forall x, y$$

(2)
$$\left.\begin{aligned} u_x &= v_y = -\cos x \cosh y \\ u_y &= -v_x = -\sin x \sinh y \end{aligned}\right\} \text{Cauchy-Riemann equations are satisfied}$$

$\Rightarrow$ by the thm of sec 22. $g' = f''(z)$ exists $\forall z$.

and $f''(z) = g'(z) = u_x + i v_x$

$= -\cos x \cosh y + i \sin x \sinh y = -f(z)$

3 (a) $f(z) = \frac{1}{z} = \frac{1}{x+iy} = \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2}$

$$u = \frac{x}{x^2+y^2} \quad , v = -\frac{y}{x^2+y^2}$$

1) $u_x = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

$u_y = \frac{-2xy}{(x^2+y^2)^2}$

$v_x = \frac{2xy}{(x^2+y^2)^2}$

$v_y = \frac{-(x^2+y^2) + 2y^2}{(x^2+y^2)} = \frac{y^2-x^2}{x^2+y^2}$

exist and continuous $\forall z \neq 0$

2) $u_x = v_y = \frac{y^2-x^2}{(x^2+y^2)^2}$

$u_y = -v_x = -\frac{2xy}{(x^2+y^2)^2}$

$\Rightarrow$ by thm of Sect 22 $f'(z)$ exists $\forall z \neq 0$

and

$f'(z) = u_x + iv_x = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{2ixy}{(x^2+y^2)^2}$

$= -\frac{\bar{z}^2}{|z|^2} = -\frac{1}{z^2}$

3 b) $f(z) = x^2 + iy^2$

$u = x^2$
$v = y^2$

(1) $\left.\begin{array}{l} u_x = 2x \\ u_y = 0 \\ v_x = 0 \\ v_y = 2y \end{array}\right\}$ exist and continuous $\forall x, y$.

(2) $\left.\begin{array}{l} u_x = v_y \Rightarrow x = y \\ u_y = -v_x = 0 \;\forall x, y \end{array}\right\} \Rightarrow x = y$ - line.

$\Rightarrow$ by thm of Sec 22 $f'(z)$
exists only along line $x = y$
($\Rightarrow z = x + ix$)

and $f'(x+ix) = u_x + i v_x = 2x$.

4 (b) $f(z) = \sqrt{r}\, e^{i\frac{\theta}{2}}$ $(r > 0,\ \alpha < \theta < \alpha + 2\pi)$

$u = \sqrt{r}\cos\frac{\theta}{2}$ $\quad v = \sqrt{r}\sin\frac{\theta}{2}$

(1) $u_r = \frac{1}{2}\frac{1}{\sqrt{r}}\cos\frac{\theta}{2}$, $v_r = \frac{1}{2}\frac{1}{\sqrt{r}}\sin\frac{\theta}{2}$ $\Big\}$

$u_\theta = -\frac{1}{2}\sqrt{r}\sin\frac{\theta}{2}$, $v_\theta = \frac{1}{2}\sqrt{r}\cos\frac{\theta}{2}$ } exist and continuous $\forall r>0, \theta$

(9)

$\Rightarrow$ $r u_r = v_\theta$, $u_\theta = -r v_r$ } these Cauchy-Riemann equations $\forall r>0$ and θ

$\Rightarrow$ by thm of Sec 23 $f'(z)$ exist $\forall r>0, \theta$

and $f'(z) = e^{-i\theta}(u_r + i v_r)$

$= e^{-i\theta}\left(\frac{1}{2}\frac{1}{\sqrt{r}}\cos\frac{\theta}{2} + i\frac{1}{2}\frac{1}{\sqrt{r}}\sin\frac{\theta}{2}\right)$

$= \frac{1}{2}\cdot\frac{1}{\sqrt{r}}e^{-i\theta}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) = \frac{1}{2}\frac{1}{\sqrt{r}}e^{-i\frac{\theta}{2}}$

$= \frac{1}{2f(z)}$

4c $f(z) = e^{-\theta}\cos(\ln r) + i e^{-\theta}\sin(\ln r)$

$r>0,\ 0<\theta<2\pi$

$u = e^{-\theta}\cos(\ln r)$, $v = e^{-\theta}\sin(\ln r)$

(a) $u_r = e^{-\theta}\frac{1}{r}(-\sin(\ln r))$ | $u_\theta = -e^{-\theta}\cos(\ln r)$ } exist and continuous $\forall r>0$.

$v_r = e^{-\theta}\frac{1}{r}\cos(\ln r)$ | $v_\theta = -e^{-\theta}\sin(\ln r)$

2)
$$\left.\begin{array}{l} r u_r = v_\theta \\ u_\theta = -r v_r \end{array}\right\} \forall r>0$$

$\Rightarrow$ by thm of Sec 23 $f'(z)$ exist $\forall r>0$

and $f'(z) = e^{-i\theta}(u_r + i v_r)$

$$= e^{-i\theta}\left(-\frac{1}{r} e^{-\theta} \sin(\ln r) + i \frac{1}{r} e^{-\theta} \cos(\ln r)\right)$$

$$= e^{-i\theta-\theta} \frac{1}{r} i \left(\cos(\ln r) + i \sin(\ln r)\right) = i \frac{e^{-i\theta-\theta}}{r} e^{i\ln r}$$

$$= \frac{i}{re^{i\theta}} f(z) = \frac{i f(z)}{z}.$$

6) $f(z) = \begin{cases} \frac{\bar{z}^2}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$

$$f(z) = \frac{\bar{z}^2}{z} = \frac{\bar{z}^3}{x^2+y^2} = \frac{x^3 + iy^3 + 3x^2(-iy) - 3y^2 x}{x^2+y^2}$$

$$= \frac{x^3 - 3y^2 x}{x^2+y^2} + i \frac{y^3 - 3x^2 y}{x^2+y^2}$$

$$u = \begin{cases} \frac{x^3 - 3y^2 x}{x^2+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases} \qquad v = \begin{cases} \frac{y^3 - 3x^2 y}{x^2+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

derivatives

$$u(x,0) = \frac{x^3}{x^2} = x$$

$$\Rightarrow u_x(x,0) = 1$$

$$u(0,y) = 0$$

$$\Rightarrow u_y(0,y) = 0$$

$$v(x,0) = 0$$

$$\Rightarrow v_x(x,0) = 0$$

$$v(0,y) = \frac{y^3}{y^2} = y$$

$$\Rightarrow v_y(0,y) = 1$$

$$\Rightarrow \quad u_x\Big|_{\substack{x=0\\y=0}} = v_y\Big|_{\substack{x=0\\y=0}} = 1$$

$$u_y\Big|_{x=0,y=0} = -v_{\cancel{x}}\Big|_{x=0,y=0} = 0$$

— Cauchy – Riemann equations are satisfied but partial derivatives are not continuous. $\Rightarrow$ $f'(0)$ does not exist.

7 $\begin{cases} u_r = u_x \cos\theta + u_y \sin\theta \\ u_\theta = -u_x r \sin\theta + u_y r \cos\theta \quad (*) \end{cases}$

$\Rightarrow$ solve for ~~v ... u_x ... v~~ -

u_x, u_y ~~assum~~ as for unknowns.

$$r\cos\theta\, u_r = u_x \cos\theta \cdot r\cos\theta + u_y \sin\theta \cos\theta\, r$$

$$+ \quad -\sin\theta\, u_\theta = +u_x r \sin^2\theta - u_y r \cos\theta \sin\theta$$

$$\Rightarrow u_x \underbrace{(\cos^2\theta + \sin^2\theta)}_{1} r = r\cos\theta\, u_r - u_\theta \sin\theta$$

$$\Rightarrow u_x = u_r \cos\theta - \frac{u_\theta}{r}\sin\theta \quad (**)$$

$\Rightarrow$ plug into $(*)$ to obtain:

$$u_y = u_r \sin\theta + u_\theta \frac{\cos\theta}{r} \quad (***)$$

Similar, $\begin{cases} v_x = v_r \cos\theta - v_\theta \frac{\sin\theta}{r} \quad (****) \\ v_y = v_r \sin\theta + v_\theta \frac{\cos\theta}{r} \quad (*****) \end{cases}$

Assume now that Cauchy-Riemann eqns. in polar form are satisfied:

$$r u_r = v_\theta \,, \quad u_\theta = -r v_r \quad (A)$$

$\Rightarrow$ plug in (A) into (**)

$$\Rightarrow \quad u_x = v_\theta \frac{\cos\theta}{r} + v_r \sin\theta = v_y \; (\text{from } (\ast\ast\ast\ast\ast))$$

and $$u_y = v_\theta \frac{\sin\theta}{r} - v_r \cos\theta = -v_x \; (\text{from } (\ast\ast\ast\ast))$$

8 (a) $f(z) = \mathrm{Re}\, z \Rightarrow \frac{\Delta w}{\Delta z} = \frac{\Delta x}{\Delta z}$

For $\Delta z = \Delta x + i0$

$\Rightarrow \frac{\Delta w}{\Delta z} = 1$

For $\Delta z = 0 + i\Delta y$

$\Rightarrow \frac{\Delta w}{\Delta z} = \frac{0}{i\Delta y} = 0$

$\Rightarrow \lim \frac{\Delta w}{\Delta z} \nexists$

$\Rightarrow f'(z)$ does not exist.

9

(a) Write $f(z) = u(r,\theta) + iv(r,\theta)$. Then recall the polar form

$$ru_r = v_\theta, \quad u_\theta = -rv_r$$

of the Cauchy-Riemann equations, which enables us to rewrite the expression (Sec. 23)

$$f'(z_0) = e^{-i\theta}(u_r + iv_r)$$

for the derivative of f at a point $z_0 = (r_0, \theta_0)$ in the following way:

$$f'(z_0) = e^{-i\theta}\left(\frac{1}{r}v_\theta - \frac{i}{r}u_\theta\right) = \frac{-i}{re^{i\theta}}(u_\theta + iv_\theta) = \frac{-i}{z_0}(u_\theta + iv_\theta).$$

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(*b*) Consider now the function

$$f(z)=\frac{1}{z}=\frac{1}{re^{i\theta}}=\frac{1}{r}e^{-i\theta}=\frac{1}{r}(\cos\theta-i\sin\theta)=\frac{\cos\theta}{r}-i\frac{\sin\theta}{r}.$$

With

$$u(r,\theta)=\frac{\cos\theta}{r} \quad\text{and}\quad v(r,\theta)=-\frac{\sin\theta}{r},$$

the final expression for $f'(z_0)$ in part (*a*) tells us that

$$f'(z)=\frac{-i}{z}\left(-\frac{\sin\theta}{r}-i\frac{\cos\theta}{r}\right)=-\frac{1}{z}\left(\frac{\cos\theta-i\sin\theta}{r}\right)$$

$$=-\frac{1}{z}\left(\frac{e^{-i\theta}}{r}\right)=-\frac{1}{z}\left(\frac{1}{re^{i\theta}}\right)=-\frac{1}{z^2}$$

when $z\neq 0$.

10. (*a*) We consider a function $F(x,y)$, where

$$x=\frac{z+\overline{z}}{2} \quad\text{and}\quad y=\frac{z-\overline{z}}{2i}.$$

Formal application of the chain rule for multivariable functions yields

$$\frac{\partial F}{\partial \overline{z}}=\frac{\partial F}{\partial x}\frac{\partial x}{\partial \overline{z}}+\frac{\partial F}{\partial y}\frac{\partial y}{\partial \overline{z}}=\frac{\partial F}{\partial x}\left(\frac{1}{2}\right)+\frac{\partial F}{\partial y}\left(-\frac{1}{2i}\right)=\frac{1}{2}\left(\frac{\partial F}{\partial x}+i\frac{\partial F}{\partial y}\right).$$

(*b*) Now define the operator

$$\frac{\partial}{\partial \overline{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right),$$

suggested by part (*a*), and formally apply it to a function $f(z)=u(x,y)+iv(x,y)$:

$$\frac{\partial f}{\partial \overline{z}}=\frac{1}{2}\left(\frac{\partial f}{\partial x}+i\frac{\partial f}{\partial y}\right)=\frac{1}{2}\frac{\partial f}{\partial x}+\frac{i}{2}\frac{\partial f}{\partial y}$$

$$=\frac{1}{2}(u_x+iv_x)+\frac{i}{2}(u_y+iv_y)=\frac{1}{2}\left[(u_x-v_y)+i(v_x+u_y)\right].$$

If the Cauchy-Riemann equations $u_x=v_y, u_y=-v_x$ are satisfied, this tells us that $\partial f/\partial \overline{z}=0$.

Homework 03

1. Assume the operators

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

and

$$\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Then proof the following Leibniz rules for complex-valued functions $F(z,\bar{z})$ and $G(z,\bar{z})$:

$$\frac{\partial}{\partial z}(FG) = G\frac{\partial F}{\partial z} + F\frac{\partial G}{\partial z},$$

$$\frac{\partial}{\partial \bar{z}}(FG) = G\frac{\partial F}{\partial \bar{z}} + F\frac{\partial G}{\partial \bar{z}}.$$

Solution

$$\frac{\partial}{\partial \bar{z}}(FG) = \frac{1}{2}(\partial_x - i\partial_y) F G = \frac{1}{2}(\partial_x F)G + \frac{1}{2}F(\partial_x G) - \frac{i}{2}(\partial_y F)G - \frac{i}{2}F\partial_y G = \left[\left(\frac{1}{2}\partial_x - \frac{i}{2}\partial_y\right)F\right]G + F\left(\frac{1}{2}\partial_x - \frac{i}{2}\partial_y\right)G = G\frac{\partial}{\partial \bar{z}}F + F\frac{\partial}{\partial \bar{z}}G.$$

Problem 01 Solution

$$\frac{\partial}{\partial \bar{z}}(FG) = \frac{1}{2}(\partial_x + i\partial_y)FG$$

$$= \frac{1}{2}(\partial_x F)G + \frac{1}{2}F(\partial_x G) + \frac{i}{2}(\partial_y F)G + \frac{i}{2}F\partial_y G$$

$$= \left(\frac{1}{2}\partial_x + \frac{i}{2}\partial_y\right)F\,G + F\left(\frac{1}{2}\partial_x + \frac{i}{2}\partial_y\right)G$$

$$= G\frac{\partial}{\partial \bar{z}}F + F\frac{\partial}{\partial \bar{z}}G.$$

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1

(a) $f(z)=\underbrace{3x+y}_{u}+i\underbrace{(3y-x)}_{v}$ is entire since

$$u_x=3=v_y \quad \text{and} \quad u_y=1=-v_x.$$

(b) $f(z)=\underbrace{\sin x\cosh y}_{u}+i\underbrace{\cos x\sinh y}_{v}$ is entire since

$$u_x=\cos x\cosh y=v_y \quad \text{and} \quad u_y=\sin x\sinh y=-v_x.$$

(c) $f(z)=e^{-y}\sin x-ie^{-y}\cos x=\underbrace{e^{-y}\sin x}_{u}+i\underbrace{(-e^{-y}\cos x)}_{v}$ is entire since

$$u_x=e^{-y}\cos x=v_y \quad \text{and} \quad u_y=-e^{-y}\sin x=-v_x.$$

(d) $f(z)=(z^2-2)e^{-x}e^{-iy}$ is entire since it is the product of the entire functions

$$g(z)=z^2-2 \quad \text{and} \quad h(z)=e^{-x}e^{-iy}=e^{-x}(\cos y-i\sin y)=\underbrace{e^{-x}\cos y}_{u}+i\underbrace{(-e^{-x}\sin y)}_{v}.$$

The function g is entire since it is a polynomial, and h is entire since

$$u_x=-e^{-x}\cos y=v_y \quad \text{and} \quad u_y=-e^{-x}\sin y=-v_x.$$

2. (a) $f(z)=\underbrace{xy}_{u}+i\underbrace{y}_{v}$ is nowhere analytic since

$$u_x=v_y \Rightarrow y=1 \quad \text{and} \quad u_y=-v_x \Rightarrow x=0,$$

which means that the Cauchy-Riemann equations hold only at the point $z=(0,1)=i$.

(b) $f(z)=2xy+i(x^2-y^2)$

$u=2xy \qquad v=x^2-y^2$

$u_x=2y \qquad v_x=2x$

$u_y=+2x \qquad v_y=-2y$

$u_x=v_y \Rightarrow 2y=-2y \Rightarrow y=0$

$u_y=-v_x \Rightarrow 2x=-2x \Rightarrow x=0$

$\Rightarrow$ Cauchy-Riemann eqns. hold only at $z=0$.

$\Rightarrow$ not analytic.

(c) $f(z)=e^y e^{ix}=e^y(\cos x+i\sin x)=\underbrace{e^y\cos x}_{u}+i\underbrace{e^y\sin x}_{v}$ is nowhere analytic since

$$u_x=v_y \Rightarrow -e^y\sin x=e^y\sin x\Rightarrow 2e^y\sin x=0\Rightarrow \sin x=0$$

and

$$u_y=-v_x \Rightarrow e^y\cos x=-e^y\cos x\Rightarrow 2e^y\cos x=0\Rightarrow \cos x=0.$$

More precisely, the roots of the equation $\sin x=0$ are $n\pi$ $(n=0,\pm1,\pm2,\ldots)$, and $\cos n\pi=(-1)^n\neq 0$. Consequently, the Cauchy-Riemann equations are not satisfied anywhere.

3

$$\frac{d\,g(f(z))}{dz}=g'(f(z))\,f'(z) \qquad (*)$$

if $g(W)$ is entire $\Rightarrow$ they both

and $f(z)$ is entire

analytic in $\mathbb{C}$ (for W and z, respectively)

$\Rightarrow$ $g(f(z))$ is analytic in $\mathbb{C}$ according to $(*)$

$\Rightarrow$ $g(f(z))$ is analytic.

$f=c_1f_1(z)+c_2f_2(z)$, f_1,f_2 entire $\Rightarrow$ differentiable in $\mathbb{C}$ $\Rightarrow$ $c_1f_2+c_2f_2$ is differentiable in $\mathbb{C}$. $\Rightarrow$ $c_1f_1+c_2f_2$ is $\boxed{\text{entire}}$.

4 (a) $f(z)=\dfrac{2z+1}{z(z^2+1)}$ – rational function

$\Rightarrow$ differentiable ~~exp~~ except $z(z^2+1)=0$

$\Rightarrow$ $z=0,\ z=\pm i$ – singular points

5

$$g(z) = \sqrt{r}\, e^{i\frac{\theta}{2}} \quad (r>0, -\pi<\theta<\pi)$$

is analytic with $g'(z) = \dfrac{1}{2g(z)}$

$$G(z) = g(2z-2+i)$$

$$\Rightarrow G'(z) = g'(2z-2+i)\,(2z-2+i)'$$

$$= \frac{1}{2g(2z-2+i)} \cdot 2 = \frac{1}{g(2z-2+i)}$$

provided $|2z-2+i|>0$ and $-\pi < Arg(2z-2+i) < \pi$.

To ensure that $|2z-2+i|>0$ and $-\pi < Arg(2z-2+i) < \pi$

we can assume e.g. that $Re(2z-2+i)>0$

$$\Rightarrow Re(2z-2+i) = 2x-2>0 \Rightarrow x>1$$

6 $g(z) = \ln r + i\theta \quad (r>0,\ 0<\theta<2\pi)$

$u = \ln r$

$v = \theta$

$$\left.\begin{array}{ll} u_r = \frac{1}{r} & v_r = 0 \\ u_\theta = 0 & v_\theta = 1 \end{array}\right\} \Rightarrow \begin{array}{l} u_r, v_r, u_\theta, v_\theta \text{ - exist} \\ \text{and continuous for } r>0, \\ 0<\theta<2\pi. \end{array}$$

$$\left.\begin{array}{l} r u_r = v_\theta = 1 \\ r v_r = -u_\theta = 0 \end{array}\right\} \Rightarrow \begin{array}{l} \text{Cauchy-Riemann eqns.} \\ \text{are satisfied} \end{array}$$

$\Rightarrow g(z)$ is analytic in $r>0$, $0<\theta<2\pi$

and $g'(z) = e^{-i\theta}(u_r + i v_r) = \dfrac{e^{-i\theta}}{r} = \dfrac{1}{z}$.

$G(z) = g(z^2+1)$

$G'(z) = g'(z^2+1)\cdot(z^2+1)' = \dfrac{1}{z^2+1}\, 2z \qquad (*)$

If $x>0$ and $y>0 \Rightarrow \operatorname{Im}(z^2+1)>0$

$\Rightarrow |z^2+1|>0$ and $-\pi < \operatorname{Arg}(z^2+1) < \pi$

$\Rightarrow G(z)$ is analytic.

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1

(a) It is straightforward to show that $u_{xx}+u_{yy}=0$ when $u(x,y)=2x(1-y)$. To find a harmonic conjugate $v(x,y)$, we start with $u_x(x,y)=2-2y$. Now

$$u_x=v_y \Rightarrow v_y=2-2y \Rightarrow v(x,y)=2y-y^2+\phi(x).$$

Then

$$u_y=-v_x \Rightarrow -2x=-\phi'(x) \Rightarrow \phi'(x)=2x \Rightarrow \phi(x)=x^2+c.$$

Consequently,

$$v(x,y)=2y-y^2+(x^2+c)=x^2-y^2+2y+c.$$

(d) It is straightforward to show that $u_{xx}+u_{yy}=0$ when $u(x,y)=\frac{y}{x^2+y^2}$. To find a harmonic conjugate $v(x,y)$, we start with $u_x(x,y)=-\frac{2xy}{(x^2+y^2)^2}$. Now

$$u_x=v_y \Rightarrow v_y=-\frac{2xy}{(x^2+y^2)^2} \Rightarrow v(x,y)=\frac{x}{x^2+y^2}+\phi(x).$$

Then

$$u_y=-v_x \Rightarrow \frac{x^2-y^2}{(x^2+y^2)^2}=\frac{x^2-y^2}{(x^2+y^2)^2}-\phi'(x) \Rightarrow \phi'(x)=0 \Rightarrow \phi(x)=c.$$

Consequently,

$$v(x,y)=\frac{x}{x^2+y^2}+c.$$

2. Suppose that v and V are harmonic conjugates of u in a domain D. This means that

$$u_x=v_y,\ u_y=-v_x \quad \text{and} \quad u_x=V_y,\ u_y=-V_x.$$

If $w=v-V$, then,

$$w_x=v_x-V_x=-u_y+u_y=0 \quad \text{and} \quad w_y=v_y-V_y=u_x-u_x=0.$$

Hence $w(x,y)=c$, where c is a (real) constant (compare the proof of the theorem in Sec. 24). That is, $v(x,y)-V(x,y)=c$.

2

5. The Cauchy-Riemann equations in polar coordinates are

$$ru_r = v_\theta \quad \text{and} \quad u_\theta = -rv_r.$$

Now

$$ru_r = v_\theta \Rightarrow ru_{rr} + u_r = v_{\theta r}$$

and

$$u_\theta = -rv_r \Rightarrow u_{\theta\theta} = -rv_{r\theta}.$$

Thus

$$r^2u_{rr} + ru_r + u_{\theta\theta} = rv_{\theta r} - rv_{r\theta};$$

and, since $v_{\theta r} = v_{r\theta}$, we have

$$r^2u_{rr} + ru_r + u_{\theta\theta} = 0,$$

which is the polar form of Laplace's equation. To show that v satisfies the same equation, we observe that

$$u_\theta = -rv_r \Rightarrow v_r = -\frac{1}{r}u_\theta \Rightarrow v_{rr} = \frac{1}{r^2}u_\theta - \frac{1}{r}u_{\theta r}$$

and

$$ru_r = v_\theta \Rightarrow v_{\theta\theta} = ru_{r\theta}.$$

Since $u_{\theta r} = u_{r\theta}$, then,

$$r^2v_{rr} + rv_r + v_{\theta\theta} = u_\theta - ru_{\theta r} - u_\theta + ru_{r\theta} = 0.$$

6. If $u(r,\theta) = \ln r$, then

$$r^2u_{rr} + ru_r + u_{\theta\theta} = r^2\left(-\frac{1}{r^2}\right) + r\left(\frac{1}{r}\right) + 0 = 0.$$

This tells us that the function $u = \ln r$ is harmonic in the domain $r > 0, 0 < \theta < 2\pi$. Now it follows from the Cauchy-Riemann equation $ru_r = v_\theta$ and the derivative $u_r = \frac{1}{r}$ that $v_\theta = 1$; thus $v(r,\theta) = \theta + \phi(r)$, where $\phi(r)$ is at present an arbitrary differentiable function of r. The other Cauchy-Riemann equation $u_\theta = -rv_r$ then becomes $0 = -r\phi'(r)$. That is, $\phi'(r) = 0$; and we see that $\phi(r) = c$, where c is an arbitrary (real) constant. Hence $v(r,\theta) = \theta + c$ is a harmonic conjugate of $u(r,\theta) = \ln r$.

HW 05 Solution

p. 92

1. (a) $\exp(2\pm3\pi i)=e^2\exp(\pm3\pi i)=-e^2$, since $\exp(\pm3\pi i)=-1$.

(b) $\exp\frac{2+\pi i}{4}=\left(\exp\frac{1}{2}\right)\left(\exp\frac{\pi i}{4}\right)=\sqrt{e}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

$$=\sqrt{e}\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)=\sqrt{\frac{e}{2}}(1+i).$$

2. $f(z)=2z^2-3-ze^z+e^{-z}$ is differentiable $\forall z$ $\Rightarrow$ it is entire.

3. First write

$$\exp(\bar{z})=\exp(x-iy)=e^xe^{-iy}=e^x\cos y-ie^x\sin y,$$

where $z=x+iy$. This tells us that $\exp(\bar{z})=u(x,y)+iv(x,y)$, where

$$u(x,y)=e^x\cos y \quad \text{and} \quad v(x,y)=-e^x\sin y.$$

Suppose that the Cauchy-Riemann equations $u_x=v_y$ and $u_y=-v_x$ are satisfied at some point $z=x+iy$. It is easy to see that, for the functions u and v here, these equations become $\cos y=0$ and $\sin y=0$. But there is no value of y satisfying this pair of equations. We may conclude that, since the Cauchy-Riemann equations fail to be satisfied anywhere, the function $\exp(\bar{z})$ is not analytic anywhere.

5. We first write

$$|\exp(2z+i)|=|\exp[2x+i(2y+1)]|=e^{2x}$$

and

$$|\exp(iz^2)|=|\exp[-2xy+i(x^2-y^2)]|=e^{-2xy}.$$

Then, since

$$|\exp(2z+i)+\exp(iz^2)|\le|\exp(2z+i)|+|\exp(iz^2)|,$$

it follows that

$$|\exp(2z+i)+\exp(iz^2)|\le e^{2x}+e^{-2xy}.$$

7. To prove that $|\exp(-2z)|<1 \Leftrightarrow \operatorname{Re} z>0$, write

$$|\exp(-2z)| = |\exp(-2x-i2y)| = \exp(-2x).$$

It is then clear that the statement to be proved is the same as $\exp(-2x)<1 \Leftrightarrow x>0$, which is obvious from the graph of the exponential function in calculus.

8. (a) Write $e^z = -2$ as $e^x e^{iy} = 2e^{i\pi}$. This tells us that

$$e^x = 2 \quad \text{and} \quad y = \pi + 2n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

That is,

$$x = \ln 2 \quad \text{and} \quad y = (2n+1)\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Hence

$$z = \ln 2 + (2n+1)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

(b) Write $e^z = 1 + \sqrt{3}i$ as $e^x e^{iy} = 2e^{i(\pi/3)}$, from which we see that

$$e^x = 2 \quad \text{and} \quad y = \frac{\pi}{3} + 2n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

That is,

$$x = \ln 2 \quad \text{and} \quad y = \left(2n + \frac{1}{3}\right)\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Consequently,

$$z = \ln 2 + \left(2n + \frac{1}{3}\right)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

10. (a) Suppose that e^z is real. Since $e^z = e^x \cos y + ie^x \sin y$, this means that $e^x \sin y = 0$. Moreover, since e^x is never zero, $\sin y = 0$. Consequently, $y = n\pi\ (n = 0, \pm 1, \pm 2, \ldots)$; that is, $\operatorname{Im} z = n\pi\ (n = 0, \pm 1, \pm 2, \ldots)$.

(b) On the other hand, suppose that e^z is pure imaginary. It follows that $\cos y = 0$, or that $y = \dfrac{\pi}{2} + n\pi\ (n = 0, \pm 1, \pm 2, \ldots)$. That is, $\operatorname{Im} z = \dfrac{\pi}{2} + n\pi\ (n = 0, \pm 1, \pm 2, \ldots)$.

14. The problem here is to establish the identity

$$(\exp z)^n = \exp(nz) \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

(a) To show that it is true when $n = 0, 1, 2, \ldots$, we use mathematical induction. It is obviously true when $n = 0$. Suppose that it is true when $n = m$, where m is any nonnegative integer. Then

$$(\exp z)^{m+1} = (\exp z)^m (\exp z) = \exp(mz)\exp z = \exp(mz + z) = \exp[(m+1)z].$$

(b) Suppose now that n is a negative integer $(n = -1, -2, \ldots)$, and write $m = -n = 1, 2, \ldots$. In view of part (a),

$$(\exp z)^n = \left(\frac{1}{\exp z}\right)^m = \frac{1}{(\exp z)^m} = \frac{1}{\exp(mz)} = \frac{1}{\exp(-nz)} = \exp(nz).$$

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1a

$$\mathrm{Log}(-ei) = \ln|-ei| + i\,\mathrm{Arg}(-ei)$$
$$= \ln e - \frac{i\pi}{2} = 1 - i\frac{\pi}{2}$$

2b

$$\log i = \ln 1 + i\left(\frac{\pi}{2} + 2\pi n\right) = \left(2n + \frac{1}{2}\right)\pi i$$
$$n = 0, \pm 1, \pm 2, \ldots$$

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(b) On the other hand,

$$\mathrm{Log}(-1+i)^2 = \mathrm{Log}(-2i) = \ln 2 - \frac{\pi}{2}i$$

and

$$2\mathrm{Log}(-1+i) = 2\left(\ln\sqrt{2} + i\frac{3\pi}{4}\right) = \ln 2 + \frac{3\pi}{2}i.$$

Hence

$$\mathrm{Log}(-1+i)^2 \neq 2\mathrm{Log}(-1+i).$$

4. (a) Consider the branch

$$\log z = \ln r + i\theta \qquad \left(r > 0, \frac{\pi}{4} < \theta < \frac{9\pi}{4}\right).$$

Since

$$\log(i^2) = \log(-1) = \ln 1 + i\pi = \pi i \quad \text{and} \quad 2\log i = 2\left(\ln 1 + i\frac{\pi}{2}\right) = \pi i,$$

we find that $\log(i^2) = 2\log i$ when this branch of $\log z$ is taken.

(7) $\log z = i\frac{\pi}{2}$ (25)

$\Rightarrow z = re^{i\theta}, \ r = 1$

$i\theta + 2\pi n = i\frac{\pi}{2}$

$\theta = \frac{\pi}{2} - 2\pi n$

$z = e^{i\frac{\pi}{2} - 2\pi i n} = i$

p.100

1. Suppose that $\operatorname{Re} z_1 > 0$ and $\operatorname{Re} z_2 > 0$. Then

$$z_1 = r_1 \exp i\Theta_1 \quad \text{and} \quad z_2 = r_2 \exp i\Theta_2,$$

where

$$-\frac{\pi}{2} < \Theta_1 < \frac{\pi}{2} \quad \text{and} \quad -\frac{\pi}{2} < \Theta_2 < \frac{\pi}{2}.$$

The fact that $-\pi < \Theta_1 + \Theta_2 < \pi$ enables us to write

$$\operatorname{Log}(z_1 z_2) = \operatorname{Log}[(r_1 r_2)\exp i(\Theta_1 + \Theta_2)] = \ln(r_1 r_2) + i(\Theta_1 + \Theta_2)$$

$$= (\ln r_1 + i\Theta_1) + (\ln r_2 + i\Theta_2) = \operatorname{Log}(r_1 \exp i\Theta_1) + \operatorname{Log}(r_2 \exp i\Theta_2)$$

$$= \operatorname{Log} z_1 + \operatorname{Log} z_2.$$

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(2)

$-\pi < \operatorname{Log} z_{1,2} \le \pi$

$\Rightarrow$

$-\pi < \operatorname{Log} z_1 + \operatorname{Log} z_2 < 2\pi$

$-\pi < \operatorname{Log}(z_1 z_2) < \pi$

$\operatorname{Log} z_1 + \operatorname{Log} z_2 = \operatorname{Log} z_1 z_2 + 2\pi i n, \quad n = 0, \pm 1, \pm 2 \ldots$

$\Rightarrow n = 0, \pm 1$

(4) $\operatorname{Log} e^{\frac{2}{3}\pi i} = \frac{2}{3}\pi i$

$\operatorname{Log} e^{-\frac{2}{3}\pi i} = -\frac{2}{3}\pi i$

$\operatorname{Log} \dfrac{e^{\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i}} = \frac{4}{3}\pi i - 2\pi i = -\frac{2}{3}\pi i$

while

$\operatorname{Log} e^{\frac{2}{3}\pi i} - \operatorname{Log} e^{-\frac{2}{3}\pi i} = -\frac{4}{3}\pi i \neq \operatorname{Log} \dfrac{e^{\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i}}$

p. 104

p. 107

28

1. In each part below, $n=0,\pm1,\pm2,\ldots$.

(a) $(1+i)^i=\exp[i\log(1+i)]=\exp\left\{i\left[\ln\sqrt{2}+i\left(\frac{\pi}{4}+2n\pi\right)\right]\right\}$

$$=\exp\left[\frac{i}{2}\ln 2-\left(\frac{\pi}{4}+2n\pi\right)\right]=\exp\left(-\frac{\pi}{4}-2n\pi\right)\exp\left(\frac{i}{2}\ln 2\right).$$

Since n takes on all integral values, the term $-2n\pi$ here can be replaced by $+2n\pi$. Thus

$$(1+i)^i=\exp\left(-\frac{\pi}{4}+2n\pi\right)\exp\left(\frac{i}{2}\ln 2\right).$$

(b) $(-1)^{1/\pi}=\exp\left[\frac{1}{\pi}\log(-1)\right]=\exp\left\{\frac{1}{\pi}[\ln 1+i(\pi+2n\pi)]\right\}=\exp[(2n+1)i].$

2. (a) P.V. $i^i=\exp(i\text{Log}\,i)=\exp\left[i\left(\ln 1+i\frac{\pi}{2}\right)\right]=\exp\left(-\frac{\pi}{2}\right).$

4a

$$(-1+\sqrt{3}i)^{3/2}=\left[(-1+\sqrt{3}\,i)^{1/2}\right]^3$$

$$-1+\sqrt{3}\,i=2e^{\frac{2\pi}{3}i}$$

$$\Rightarrow (-1+\sqrt{3}\,i)^{1/2}=\pm\sqrt{2}\,e^{\frac{\pi i}{3}}$$

$$\Rightarrow \left[(-1+\sqrt{3}i)^{1/2}\right]^3=\pm(\sqrt{2})^3\,e^{\pi i}$$

$$=\pm 2\sqrt{2}$$

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$$\cancel{e^{f(z)}}\quad (c^{f(z)})'=(e^{f(z)\log c})'=f'(z)\log c\, e^{f(z)}$$