math 561 Solutions HW 03

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1. The desired derivatives can be found by writing

$$\frac{d}{dz}\sin z = \frac{d}{dz}\left(\frac{e^{iz}-e^{-iz}}{2i}\right) = \frac{1}{2i}\left(\frac{d}{dz}e^{iz} - \frac{d}{dz}e^{-iz}\right)$$
$$= \frac{1}{2i}\left(ie^{iz} + ie^{-iz}\right) = \frac{e^{iz}+e^{-iz}}{2} = \cos z$$

and

$$\frac{d}{dz}\cos z = \frac{d}{dz}\left(\frac{e^{iz}+e^{-iz}}{2}\right) = \frac{1}{2}\left(\frac{d}{dz}e^{iz} + \frac{d}{dz}e^{-iz}\right)$$
$$= \frac{1}{2}\left(ie^{iz} - ie^{-iz}\right)\cdot\frac{i}{i} = -\frac{e^{iz}-e^{-iz}}{2i} = -\sin z.$$

2a

$$e^{iz_1} e^{iz_2} = (\cos z_1 + i\sin z_1)(\cos z_2 + i \sin z_2)$$

$$= \cos z_1 \cos z_2 - \sin z_1 \sin z_2 + i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2)$$

But know $\cos(-z) = \cos z$ and $\sin(-z) = -\sin z$

$$\Rightarrow e^{-iz_1} e^{-iz_2} = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 - i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2)$$

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3. We know from Exercise 2(*b*) that

$$\sin(z+z_2)=\sin z\cos z_2+\cos z\sin z_2.$$

Differentiating each side yields

$$\cos(z+z_2)=\cos z\cos z_2-\sin z\sin z_2.$$

Then, by setting $z=z_1$, we have

$$\cos(z_1+z_2)=\cos z_1\cos z_2-\sin z_1\sin z_2.$$

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$$\frac{d}{dz}\tan z=\frac{d}{dz}\frac{\sin z}{\cos z}=\frac{\cos^2 z+\sin^2 z}{\cos^2 z}=\frac{1}{\cos^2 z}$$

$$\frac{d}{dz}\sec z=\frac{d}{dz}\frac{1}{\cos z}=\frac{\sin z}{\cos^2 z}=\sec z\tan z.$$

11. By writing $f(z) = \sin\overline{z} = \sin(x - iy) = \sin x \cosh y - i\cos x \sinh y$, we have

$$f(z) = u(x,y) + iv(x,y),$$

where

$$u(x,y) = \sin x \cosh y \quad \text{and} \quad v(x,y) = -\cos x \sinh y.$$

If the Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$ are to hold, it is easy to see that

$$\cos x \cosh y = 0 \quad \text{and} \quad \sin x \sinh y = 0.$$

Since $\cosh y$ is never zero, it follows from the first of these equations that $\cos x = 0$; that is, $x = \frac{\pi}{2} + n\pi$ $(n = 0 \pm 1, \pm 2, \ldots)$. Furthermore, since $\sin x$ is nonzero for each of these values of x, the second equation tells us that $\sinh y = 0$, or $y = 0$. Thus the Cauchy-Riemann equations hold only at the points

$$z = \frac{\pi}{2} + n\pi \qquad (n = 0 \pm 1, \pm 2, \ldots).$$

Evidently, then, there is no neighborhood of any point throughout which f is analytic, and we may conclude that $\sin\overline{z}$ is not analytic anywhere.

The function $f(z) = \cos\overline{z} = \cos(x - iy) = \cos x \cosh y + i\sin x \sinh y$ can be written as

$$f(z) = u(x,y) + iv(x,y),$$

where

$$u(x,y) = \cos x \cosh y \quad \text{and} \quad v(x,y) = \sin x \sinh y.$$

If the Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$ hold, then

$$\sin x \cosh y = 0 \quad \text{and} \quad \cos x \sinh y = 0.$$

The first of these equations tells us that $\sin x = 0$, or $x = n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$. Since $\cos n\pi \neq 0$, it follows that $\sinh y = 0$, or $y = 0$. Consequently, the Cauchy-Riemann equations hold only when

$$z = n\pi \qquad (n = 0 \pm 1, \pm 2, \ldots).$$

So there is no neighborhood throughout which f is analytic, and this means that $\cos\overline{z}$ is nowhere analytic.

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1. To find the derivatives of sinh$z$ and cosh$z$, we write

$$\frac{d}{dz}\sinh z = \frac{d}{dz}\left(\frac{e^z - e^{-z}}{2}\right) = \frac{1}{2}\frac{d}{dz}(e^z - e^{-z}) = \frac{e^z + e^{-z}}{2} = \cosh z$$

and

$$\frac{d}{dz}\cosh z = \frac{d}{dz}\left(\frac{e^z + e^{-z}}{2}\right) = \frac{1}{2}\frac{d}{dz}(e^z + e^{-z}) = \frac{e^z - e^{-z}}{2} = \sinh z.$$

3. Identity (9), Sec. 34, is $\sin^2 z + \cos^2 z = 1$. Replacing z by iz here and using the identities

$$\sin(iz) = i\sinh z \quad \text{and} \quad \cos(iz) = \cosh z,$$

we find that $i^2 \sinh^2 z + \cosh^2 z = 1$, or

$$\cosh^2 z - \sinh^2 z = 1.$$

Identity (6), Sec. 34, is $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$. Replacing z_1 by iz_1 and z_2 by iz_2 here, we have $\cos[i(z_1 + z_2)] = \cos(iz_1)\cos(iz_2) - \sin(iz_1)\sin(iz_2)$. The same identities that were used just above then lead to

$$\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2.$$

7. *(a)* Observe that

$$\sinh(z + \pi i) = \frac{e^{z+\pi i} - e^{-(z+\pi i)}}{2} = \frac{e^z e^{\pi i} - e^{-z}e^{-\pi i}}{2} = \frac{-e^z + e^{-z}}{2} = -\frac{e^z - e^{-z}}{2} = -\sinh z.$$

(b) Also,

$$\cosh(z + \pi i) = \frac{e^{z+\pi i} + e^{-(z+\pi i)}}{2} = \frac{e^z e^{\pi i} + e^{-z}e^{-\pi i}}{2} = \frac{-e^z - e^{-z}}{2} = -\frac{e^z + e^{-z}}{2} = -\cosh z.$$

(c) From parts *(a)* and *(b)*, we find that

$$\tanh(z + \pi i) = \frac{\sinh(z + \pi i)}{\cosh(z + \pi i)} = \frac{-\sinh z}{-\cosh z} = \frac{\sinh z}{\cosh z} = \tanh z.$$

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a $\tan^{-1}(2i) = \frac{i}{2}\log\frac{i+2i}{i-2i}$

$= \frac{i}{2}\log\frac{3i}{-i} = \frac{i}{2}\log(-3)$

$= \frac{i}{2}\left[\ln 3 - i\pi + 2\pi i n\right]$

$= (n+\frac{1}{2})\pi + \frac{i}{2}\ln 3, \quad n = 0, \pm 1, \ldots$

c $\cosh^{-1}(-1) = \log\left[-1 + (1-1)^{1/2}\right]$

$= (2n+1)\pi i \quad n = 0, \pm 1, \ldots$

2 a $\sin z = 2$

$z = x + iy$

$\sin(x+iy) = \sin x \cos iy + \cos x \sin iy$

$= \sin x \cosh y + i \cos x \sinh y = 2$

$\Rightarrow \cos x = 0 \Rightarrow x = \pi n + \frac{\pi}{2} \quad n = 0, \pm 1 \ldots$

$\sin x \cosh y = 2$

$(-1)^n$

$\Rightarrow \cosh y = (-1)^n 2$ but $\cosh y > 0 \ \forall y$

$\Rightarrow n = 2m$

$\cosh y = 2$

$e^y + e^{-y} = 4$

$e^{2y} - 4e^y + 1 = 0$

$e^y = 2 \pm \sqrt{3}$

$y = \ln(2 \pm \sqrt{3}) = \pm\ln(2+\sqrt{3})$

$\left(\text{because } \ln(2-\sqrt{3}) = \ln\left(\frac{4-3}{2+\sqrt{3}}\right) = -\ln(2+\sqrt{3}) \right)$

$\Rightarrow z = (2m + \frac{1}{2})\pi \pm i\ln(2+\sqrt{3}),$

$m = 0, \pm 1, \pm 2, \ldots$

(16)

$\sin^{-1} z = -i \log[iz + (1-4)^{1/2}]$

$= -i \log(2i \pm i\sqrt{3}) = -i\left[\pm\ln(2+\sqrt{3}) + \frac{\pi}{2} + 2\pi n\right]$

$= \left(\frac{1}{2} + 2n\right)\pi \pm i\ln(2+\sqrt{3}), \quad n = 0, \pm 1, \pm 2 \ldots$

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1 (a) $w(t) = u(t) + i v(t)$

$$\frac{d}{dt} w(-t) = \frac{d}{dt} u(-t) + i \frac{d}{dt} v(-t)$$

$$= -u'(-t) - i v'(-t) = -w'(-t)$$

(b) $$\frac{d}{dt}[w(t)]^2 = \left(u^2(t) - v^2(t) + 2i u(t) v(t)\right)'$$

$$= 2uu' - 2vv' + 2iu'v + 2iuv'$$

$$= 2(u+iv)(u'+iv') = 2ww'.$$

2. (a) $\int_1^2 \left(\frac{1}{t} - i\right)^2 dt = \int_1^2 \left(\frac{1}{t^2} - 1\right) dt - 2i \int_1^2 \frac{dt}{t} = -\frac{1}{2} - 2i \ln 2 = -\frac{1}{2} - i \ln 4;$

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(c) Since $|e^{-bz}|=e^{-bx}$, we find that

$$\int_0^\infty e^{-zt}dt=\lim_{b\to\infty}\int_0^b e^{-zt}dt=\lim_{b\to\infty}\left[\frac{e^{-zt}}{-z}\right]_{t=0}^{t=b}=\frac{1}{z}\lim_{b\to\infty}\left(1-e^{-bz}\right)=\frac{1}{z} \quad \text{when } \operatorname{Re} z>0.$$

3. The problem here is to verify that

$$\int_0^{2\pi} e^{im\theta}e^{-in\theta}d\theta=\begin{cases}0 & \text{when } m\neq n,\\ 2\pi & \text{when } m=n.\end{cases}$$

To do this, we write

$$I=\int_0^{2\pi} e^{im\theta}e^{-in\theta}d\theta=\int_0^{2\pi} e^{i(m-n)\theta}d\theta$$

and observe that when $m\neq n$,

$$I=\left[\frac{e^{i(m-n)\theta}}{i(m-n)}\right]_0^{2\pi}=\frac{1}{i(m-n)}-\frac{1}{i(m-n)}=0.$$

When $m=n$, I becomes

$$I=\int_0^{2\pi} d\theta=2\pi;$$

and the verification is complete.

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1. (a) Start by writing

$$I=\int_{-b}^{-a} w(-t)dt=\int_{-b}^{-a} u(-t)dt+i\int_{-b}^{-a} v(-t)dt.$$

The substitution $\tau=-t$ in each of these two integrals on the right then yields

$$I=-\int_{b}^{a} u(\tau)d\tau-i\int_{b}^{a} v(\tau)d\tau=\int_{a}^{b} u(\tau)d\tau+i\int_{a}^{b} v(\tau)d\tau=\int_{a}^{b} w(\tau)d\tau.$$

That is,

$$\int_{-b}^{-a} w(-t)dt=\int_{a}^{b} w(\tau)d\tau.$$

(b) Start with

$$I=\int_{a}^{b} w(t)dt=\int_{a}^{b} u(t)dt+i\int_{a}^{b} v(t)dt$$

and then make the substitution $t=\varphi(\tau)$ in each of the integrals on the right. The result is

$$I=\int_{\alpha}^{\beta} u[\phi(\tau)]\phi'(\tau)d\tau+i\int_{\alpha}^{\beta} v[\phi(\tau)]\phi'(\tau)d\tau=\int_{\alpha}^{\beta} w[\phi(\tau)]\phi'(\tau)d\tau.$$

That is,

$$\int_{a}^{b} w(t)dt=\int_{\alpha}^{\beta} w[\phi(\tau)]\phi'(\tau)d\tau.$$

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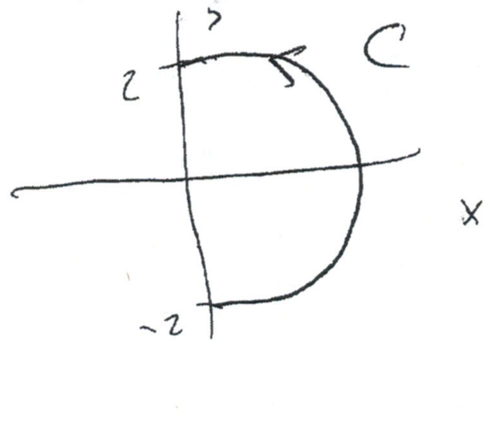

(a) $z = \cancel{2e}\,1 = 2e^{i\theta}$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

(8) $z = z(y) = \sqrt{4-y^2} + iy \quad , \quad -2 \le y \le 2$

$$z(\varphi(y)) = 2e^{i\theta} = 2e^{i \arctan \frac{y}{\sqrt{4-y^2}}}$$

$$\theta = \arctan \frac{y}{\sqrt{4-y^2}}$$

$$\cos y = \frac{1}{\sqrt{\tan^2 y + 1}}$$

$$\sin y = \frac{\tan y}{\sqrt{\tan^2 y + 1}}$$

$$\Rightarrow 2e^{i \arctan \frac{y}{\sqrt{4-y^2}}} = 2\left[\frac{1}{\sqrt{\frac{y^2}{4-y^2}+1}} + \frac{i\frac{y}{\sqrt{4-y^2}}}{\sqrt{\frac{y^2}{4-y^2}+1}}\right]$$

$$= 2\left[\frac{\sqrt{4-y^2}}{2} + \frac{iy}{2}\right] = \sqrt{4-y^2} + iy$$

$$-2 \le y \le 2$$

$$\varphi'(y) = \left(\arctan \frac{y}{\sqrt{4-y^2}}\right)' = \frac{1}{1+\frac{y^2}{4-y^2}} \cdot \frac{1\cdot\sqrt{4-y^2} - \frac{y(-2y)}{2\sqrt{4-y^2}}}{4-y^2}$$

$$= \frac{1}{4}\left(\frac{4-y^2+y^2}{\sqrt{4-y^2}}\right) = \frac{1}{4}\frac{4}{\sqrt{4-y^2}} > 0$$

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5. If $w(t)=f[z(t)]$ and $f(z)=u(x,y)+iv(x,y)$, $z(t)=x(t)+iy(t)$, we have

$$w(t)=u[x(t),y(t)]+iv[x(t),y(t)].$$

The chain rule tells us that

$$\frac{du}{dt}=u_x x'+u_y y' \quad \text{and} \quad \frac{dv}{dt}=v_x x'+v_y y',$$

and so

$$w'(t)=(u_x x'+u_y y')+i(v_x x'+v_y y').$$

In view of the Cauchy-Riemann equations $u_x=v_y$ and $u_y=-v_x$, then,

$$w'(t)=(u_x x'-v_x y')+i(v_x x'+u_x y')=(u_x+iv_x)(x'+iy').$$

That is,

$$w'(t)=\{u_x[x(t),y(t)]+iv_x[x(t),y(t)]\}[x'(t)+iy'(t)]=f'[z(t)]z'(t)$$

when $t=t_0$.

6.

$$0 \le x \le 1$$

$$y(x)=\begin{cases} x^3 \sin\frac{\pi}{x} & , 0<x\le 1 \\ 0 & , x=0 \end{cases}$$

(a) $z = x + i y(x)$

$$y(x)=0 \Rightarrow x^3 \sin\frac{\pi}{x}=0$$
$$\Rightarrow \frac{\pi}{x} = \pi n, \quad n=1,2,\ldots$$
$$\Rightarrow x = \frac{1}{n}, \quad n=1,2,\ldots$$

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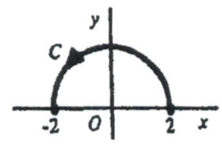
1. (a) Let C be the semicircle $z=2e^{i\theta}$ $(0\le\theta\le\pi)$, shown below.

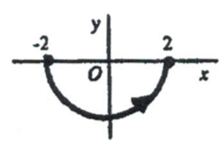


Then

$$\int_C \frac{z+2}{z}dz = \int_C\left(1+\frac{2}{z}\right)dz = \int_0^{\pi}\left(1+\frac{2}{2e^{i\theta}}\right)2ie^{i\theta}d\theta = 2i\int_0^{\pi}(e^{i\theta}+1)d\theta$$

$$= 2i\left[\frac{e^{i\theta}}{i}+\theta\right]_0^{\pi} = 2i(i+\pi+i) = -4+2\pi i.$$

(b) Now let C be the semicircle $z=2e^{i\theta}$ $(\pi\le\theta\le 2\pi)$ just below.



This is the same as part (a), except for the limits of integration. Thus

$$\int_C \frac{z+2}{z}dz = 2i\left[\frac{e^{i\theta}}{i}+\theta\right]_{\pi}^{2\pi} = 2i(-i+2\pi-i-\pi) = 4+2\pi i.$$

(c) Finally, let C denote the entire circle $z=2e^{i\theta}$ $(0\le\theta\le 2\pi)$. In this case,

$$\int_C \frac{z+2}{z}dz = 4\pi i,$$

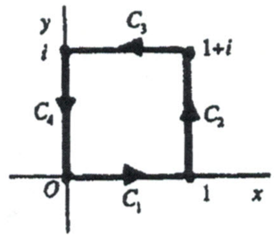
the value here being the sum of the values of the integrals in parts (a) and (b).

2. (a) The arc is $C: z=1+e^{i\theta}$ $(\pi\le\theta\le 2\pi)$. Then

$$\int_C (z-1)dz = \int_{\pi}^{2\pi}(1+e^{i\theta}-1)ie^{i\theta}d\theta = i\int_{\pi}^{2\pi}e^{i2\theta}d\theta = i\left[\frac{e^{i2\theta}}{2i}\right]_{\pi}^{2\pi}$$

$$= \frac{1}{2}\left(e^{i4\pi}-e^{i2\pi}\right) = \frac{1}{2}(1-1) = 0.$$

3. In this problem, the path C is the sum of the paths C_1, C_2, C_3, and C_4 that are shown below.



The function to be integrated around the closed path C is $f(z)=\pi e^{\pi \bar{z}}$. We observe that $C=C_1+C_2+C_3+C_4$ and find the values of the integrals along the individual legs of the square C.

(i) Since C_1 is $z=x\,(0\leq x\leq 1)$,

$$\int_{C_1}\pi e^{\pi \bar{z}}dz=\pi\int_0^1 e^{\pi x}dx=e^{\pi}-1.$$

(ii) Since C_2 is $z=1+iy\,(0\leq y\leq 1)$,

$$\int_{C_2}\pi e^{\pi \bar{z}}dz=\pi\int_0^1 e^{\pi(1-iy)}i\,dy=e^{\pi}\pi i\int_0^1 e^{-i\pi y}dy=2e^{\pi}.$$

(iii) Since C_3 is $z=(1-x)+i\,(0\leq x\leq 1)$,

$$\int_{C_3}\pi e^{\pi \bar{z}}dz=\pi\int_0^1 e^{\pi[(1-x)-i]}(-1)dx=\pi e^{\pi}\int_0^1 e^{-\pi x}dx=e^{\pi}-1.$$

(iv) Since C_4 is $z=i(1-y)\,(0\leq y\leq 1)$,

$$\int_{C_4}\pi e^{\pi \bar{z}}dz=\pi\int_0^1 e^{-\pi(1-y)i}(-i)dy=\pi i\int_0^1 e^{i\pi y}dy=-2.$$

Finally, then, since

$$\int_C \pi e^{\pi \bar{z}}dz=\int_{C_1}\pi e^{\pi \bar{z}}dz+\int_{C_2}\pi e^{\pi \bar{z}}dz+\int_{C_3}\pi e^{\pi \bar{z}}dz+\int_{C_4}\pi e^{\pi \bar{z}}dz,$$

we find that

$$\int_C \pi e^{\pi \bar{z}}dz=4(e^{\pi}-1).$$

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4. The path C is the sum of the paths

$$C_1: z = x + ix^3\ (-1 \le x \le 0) \quad \text{and} \quad C_2: z = x + ix^3\ (0 \le x \le 1).$$

Using

$$f(z) = 1 \text{ on } C_1 \quad \text{and} \quad f(z) = 4y = 4x^3 \text{ on } C_2,$$

we have

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz = \int_{-1}^{0} 1(1+i3x^2)dx + \int_0^1 4x^3(1+i3x^2)dx$$

$$= \int_{-1}^{0} dx + 3i\int_{-1}^{0} x^2 dx + 4\int_0^1 x^3 dx + 12i\int_0^1 x^5 dx$$

$$= \left[x\right]_{-1}^{0} + i\left[x^3\right]_{-1}^{0} + \left[x^4\right]_0^1 + 2i\left[x^6\right]_0^1 = 1 + i + 1 + 2i = 2 + 3i.$$

8. Let C be the positively oriented circle $|z|=1$, with parametric representation $z = e^{i\theta}\ (0 \le \theta \le 2\pi)$, and let m and n be integers. Then

$$\int_C z^m \overline{z}^n dz = \int_0^{2\pi} \left(e^{i\theta}\right)^m \left(e^{-i\theta}\right)^n ie^{i\theta} d\theta = i\int_0^{2\pi} e^{i(m+1)\theta} e^{-in\theta} d\theta.$$

But we know from Exercise 3, Sec. 38, that

$$\int_0^{2\pi} e^{im\theta} e^{-in\theta} d\theta = \begin{cases} 0 & \text{when} \quad m \ne n, \\ 2\pi & \text{when} \quad m = n. \end{cases}$$

Consequently,

$$\int_C z^m \overline{z}^n dz = \begin{cases} 0 & \text{when} \quad m+1 \ne n, \\ 2\pi i & \text{when} \quad m+1 = n. \end{cases}$$

11. (a) The function $f(z)$ is continuous on a smooth arc C, which has a parametric representation $z = z(t)\ (a \le t \le b)$. Exercise 1(b), Sec. 38, enables us to write

$$\int_a^b f[z(t)]z'(t)dt = \int_\alpha^\beta f[Z(\tau)]z'[\phi(\tau)]\phi'(\tau)d\tau,$$

where

$$Z(\tau) = z[\phi(\tau)] \qquad (\alpha \le \tau \le \beta).$$

But expression (14), Sec. 38, tells us that

$$z'[\phi(\tau)]\phi'(\tau) = Z'(\tau);$$

and so

$$\int_a^b f[z(t)]z'(t)dt = \int_\alpha^\beta f[Z(\tau)]Z'(\tau)d\tau.$$

(b) Suppose that C is any contour and that $f(z)$ is piecewise continuous on C. Since C can be broken up into a finite chain of smooth arcs on which $f(z)$ is continuous, the identity obtained in part *(a)* remains valid.

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6. To integrate the branch

$$z^{-1+i} = e^{(-1+i)\log z} \qquad (|z|>0, 0<\arg z<2\pi)$$

around the circle $C: z = e^{i\theta}$ $(0 \le \theta \le 2\pi)$, write

$$\int_C z^{-1+i}\,dz = \int_C e^{(-1+i)\log z}\,dz = \int_0^{2\pi} e^{(-1+i)(\ln 1+i\theta)}\, ie^{i\theta}\,d\theta = i\int_0^{2\pi} e^{-i\theta-\theta}\, e^{i\theta}\,d\theta = i\int_0^{2\pi} e^{-\theta}\,d\theta = i\left(1-e^{-2\pi}\right).$$

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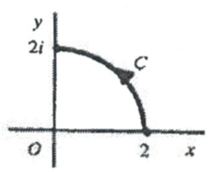
1. Let C be the arc of the circle $|z|=2$ shown below.



Without evaluating the integral, let us find an upper bound for $\left|\int_C \frac{dz}{z^2-1}\right|$. To do this, we note that if z is a point on C,

$$\left|z^2-1\right| \ge \left||z^2|-1\right| = \left||z|^2-1\right| = |4-1| = 3.$$

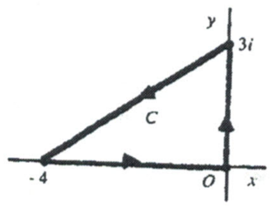
Thus

$$\left|\frac{1}{z^2-1}\right| = \frac{1}{|z^2-1|} \le \frac{1}{3}.$$

Also, the length of C is $\frac{1}{4}(4\pi) = \pi$. So, taking $M = \frac{1}{3}$ and $L = \pi$, we find that

$$\left|\int_C \frac{dz}{z^2-1}\right| \le ML = \frac{\pi}{3}.$$

3. The contour C is the closed triangular path shown below.



To find an upper bound for $\left|\int_C (e^z - \bar{z})\,dz\right|$, we let z be a point on C and observe that

$$|e^z - \bar{z}| \leq |e^z| + |\bar{z}| = e^x + \sqrt{x^2 + y^2}.$$

But $e^x \leq 1$ since $x \leq 0$, and the distance $\sqrt{x^2+y^2}$ of the point z from the origin is always less than or equal to 4. Thus $|e^z - \bar{z}| \leq 5$ when z is on C. The length of C is evidently 12. Hence, by writing $M = 5$ and $L = 12$, we have

$$\left|\int_C (e^z - \bar{z})\,dz\right| \leq ML = 60.$$

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(1.) The function z^n $(n = 0,1,2,\ldots)$ has the antiderivative $z^{n+1}/(n+1)$ everywhere in the finite plane. Consequently, for any contour C from a point z_1 to a point z_2,

$$\int_C z^n dz = \int_{z_1}^{z_2} z^n dz = \left[\frac{z^{n+1}}{n+1}\right]_{z_1}^{z_2} = \frac{z_2^{n+1}}{n+1} - \frac{z_1^{n+1}}{n+1} = \frac{1}{n+1}\left(z_2^{n+1} - z_1^{n+1}\right).$$

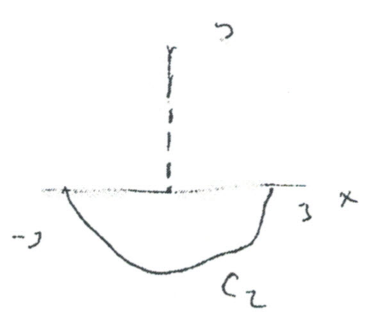
(2.) (a) $$\int_i^{i/2} e^{\pi z} dz = \left[\frac{e^{\pi z}}{\pi}\right]_i^{i/2} = \frac{e^{i\pi/2} - e^{i\pi}}{\pi} = \frac{i+1}{\pi} = \frac{1+i}{\pi}.$$

(b) $$\int_0^{\pi+2i} \cos\left(\frac{z}{2}\right) dz = 2\left[\sin\left(\frac{z}{2}\right)\right]_0^{\pi+2i} = 2\sin\left(\frac{\pi}{2}+i\right) = 2\,\frac{e^{i\left(\frac{\pi}{2}+i\right)} - e^{-i\left(\frac{\pi}{2}+i\right)}}{2i} = -i\left(e^{i\pi/2}e^{-1} - e^{-i\pi/2}e\right)$$

$$= -i\left(\frac{i}{e} + ie\right) = \frac{1}{e} + e = e + \frac{1}{e}.$$

(c) $$\int_1^3 (z-2)^3 dz = \left[\frac{(z-2)^4}{4}\right]_1^3 = \frac{1}{4} - \frac{1}{4} = 0.$$

(4) $f_2(z) = r e^{i\theta/2}$ $\quad r>0, \ \frac{\pi}{2} < \theta < \frac{5}{2}\pi$



$$\int_{C_2} z^{1/2} dz$$

$$F_2(z) = \frac{2}{3} r\sqrt{r}\, e^{i\frac{3\theta}{2}}, \quad \frac{\pi}{2} < \theta < \frac{5\pi}{2}$$

$$\int_{C_2} z^{1/2} dz = \frac{2}{3} 3\sqrt{3}\, e^{i\frac{3\cdot 2\pi}{2}} - \frac{2}{3} 3\sqrt{3}\, e^{i\frac{3\pi}{2}}$$

$$= 2\sqrt{3}\left(-1 - (-i)\right) = 2\sqrt{3}(-1+i)$$

5. Let C denote any contour from $z=-1$ to $z=1$ that, except for its end points, lies above the real axis. This exercise asks us to evaluate the integral

$$I=\int_{-1}^{1} z^i dz,$$

where z^i denotes the principal branch

$$z^i=\exp(i\,\mathrm{Log}\,z) \qquad (|z|>0,\ -\pi<\mathrm{Arg}\,z<\pi).$$

An antiderivative of this branch *cannot* be used since the branch is not even defined at $z=-1$. But the integrand can be replaced by the branch

$$z^i=\exp(i\log z) \qquad \left(|z|>0,\ -\frac{\pi}{2}<\arg z<\frac{3\pi}{2}\right)$$

since it agrees with the integrand along C. Using an antiderivative of this new branch, we can now write

$$I=\left[\frac{z^{i+1}}{i+1}\right]_{-1}^{1}=\frac{1}{i+1}\left[(1)^{i+1}-(-1)^{i+1}\right]=\frac{1}{i+1}\left[e^{(i+1)\log 1}-e^{(i+1)\log(-1)}\right]$$

$$=\frac{1}{i+1}\left[e^{(i+1)(\ln 1+i0)}-e^{(i+1)(\ln 1+i\pi)}\right]=\frac{1}{i+1}\left(1-e^{-\pi}e^{i\pi}\right)=\frac{1+e^{-\pi}}{1+i}\cdot\frac{1-i}{1-i}$$

$$=\frac{1+e^{-\pi}}{2}(1-i).$$