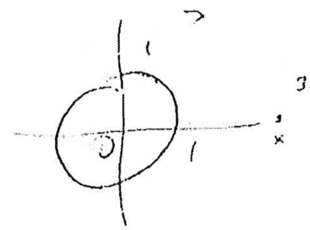
math 561 HW 05 Solutions

1

p.160 (1)

(a)



$f(z) = \frac{z^2}{z-3}$ Function is analytic everywhere except $z=3$

$\Rightarrow$ analytic inside and on $|z|=1$ circle

$\Rightarrow$ by Cauchy - Goursat thm

$$\int_C \frac{z^2}{z-3}\, dz = 0$$

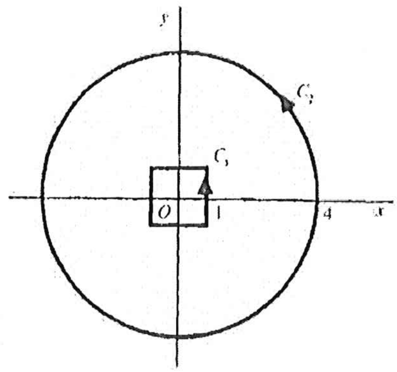
(c) $f(z) = \frac{1}{z^2+2z+2} = \frac{1}{(z+1+i)(z+1-i)}$

$z = -1 \pm \sqrt{1-2} = -1 \pm i$

$f(z)$ is analytic everywhere except $z = -1 \pm i$ $\Rightarrow$ inside and on C_0 $f(z)$ is analytic

$$\Rightarrow \int_C \frac{dz}{z^2+2z+2} = 0$$

2. The contours C_1 and C_2 are as shown in the figure below.

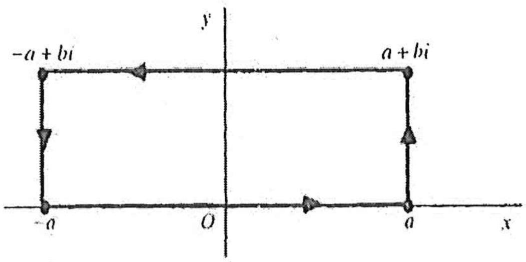


In each of the cases below, the singularities of the integrand lie inside C_1 or outside of C_2; and so the integrand is analytic on the contours and between them. Consequently,

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

(b) When $f(z) = \dfrac{z+2}{\sin(z/2)}$, the singularities are at $z = 2n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$.

4. (a) In order to derive the integration formula in question, we integrate the function e^{-z^2} around the closed rectangular path shown below.



Since the lower horizontal leg is represented by $z = x$ $(-a \le x \le a)$, the integral of e^{-z^2} along that leg is

$$\int_{-a}^{a} e^{-x^2}\,dx = 2\int_{0}^{a} e^{-x^2}\,dx.$$

Since the opposite direction of the upper horizontal leg has parametric representation $z = x + bi$ $(-a \le x \le a)$, the integral of e^{-z^2} along the upper leg is

$$-\int_{-a}^{a} e^{-(x+bi)^2}\,dx = -e^{b^2}\int_{-a}^{a} e^{-x^2}e^{-i2bx}\,dx = -e^{b^2}\int_{-a}^{a} e^{-x^2}\cos 2bx\,dx + ie^{b^2}\int_{-a}^{a} e^{-x^2}\sin 2bx\,dx,$$

or simply

$$-2e^{b^2}\int_{0}^{a} e^{-x^2}\cos 2bx\,dx.$$

Since the right-hand vertical leg is represented by $z = a + iy$ $(0 \le y \le b)$, the integral of e^{-z^2} along it is

$$\int_{0}^{b} e^{-(a+iy)^2}\,i\,dy = ie^{-a^2}\int_{0}^{b} e^{y^2}e^{-i2ay}\,dy.$$

Finally, since the opposite direction of the left-hand vertical leg has the representation $z = -a + iy$ $(0 \le y \le b)$, the integral of e^{-z^2} along that vertical leg is

$$-\int_{0}^{b} e^{-(-a+iy)^2}\,i\,dy = -ie^{-a^2}\int_{0}^{b} e^{y^2}e^{i2ay}\,dy.$$

According to the Cauchy-Goursat theorem, then,

$$2\int_{0}^{a} e^{-x^2}\,dx - 2e^{b^2}\int_{0}^{a} e^{-x^2}\cos 2bx\,dx + ie^{-a^2}\int_{0}^{b} e^{y^2}e^{-i2ay}\,dy - ie^{-a^2}\int_{0}^{b} e^{y^2}e^{i2ay}\,dy = 0;$$

and this reduces to

$$\int_{0}^{a} e^{-x^2}\cos 2bx\,dx = e^{-b^2}\int_{0}^{a} e^{-x^2}\,dx + e^{-(a^2+b^2)}\int_{0}^{b} e^{y^2}\sin 2ay\,dy.$$

(*b*) We now let $a \to \infty$ in the final equation in part (*a*), keeping in mind the known integration formula

$$\int_{0}^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

and the fact that

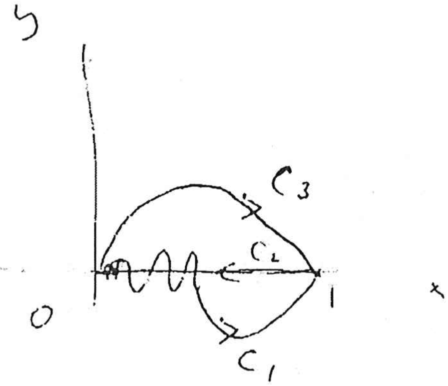
$$\left| e^{-(a^2+b^2)}\int_{0}^{b} e^{y^2}\sin 2ay\,dy \right| \le e^{-(a^2+b^2)}\int_{0}^{b} e^{y^2}\,dy \to 0 \text{ as } a \to \infty.$$

The result is

$$\int_{0}^{\infty} e^{-x^2}\cos 2bx\,dx = \frac{\sqrt{\pi}}{2}e^{-b^2} \qquad (b > 0).$$

5.

$$y(x) = \begin{cases} x^3 \sin\frac{\pi}{x}, & 0 < x \le 1 \\ 0, & x = 0 \end{cases}$$



C_2 connects 0 and 1

C_1, C_2, C_3 – smooth arcs

$C_1 - C_3$ – simple closed contour

(because $z'(t)$ is piecewise continuous)

the same for

$C_2 + C_3$

$$\Rightarrow \int_{C_1} f(z)\,dz + \int_{-C_3} f(z)\,dz = 0 \Rightarrow \int_{C_1} f(z)\,dz = \int_{C_3} f(z)\,dz$$

$$\int_{C_2} f(z)\,dz + \int_{C_3} f(z)\,dz = 0$$ by

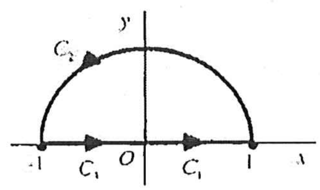
Cauchy-Goursat thm.

$$\int_C f(z)dz = \int_{C_1+C_2} f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$$

$$\int_{C_3} f(z)dz$$

$$= \left(\int_{C_3} + \int_{C_2}\right) f(z)dz = 0$$

6. We let C denote the entire boundary of the semicircular region appearing below. It is made up of the leg C_1 from the origin to the point $z = 1$, the semicircular arc C_2 that is shown, and the leg C_3 from $z = -1$ to the origin. Thus $C = C_1 + C_2 + C_3$.



We also let $f(z)$ be a continuous function that is defined on this closed semicircular region by writing $f(0) = 0$ and using the branch

$$f(z) = \sqrt{r}e^{i\theta/2} \qquad \left(r > 0, -\frac{\pi}{2} < \theta < \frac{3\pi}{2}\right)$$

of the multiple-valued function $z^{1/2}$. The problem here is to evaluate the integral of $f(z)$ around C by evaluating the integrals along the individual paths C_1, C_2, and C_3 and then adding the results. In each case, we write a parametric representation for the path (or a related one) and then use it to evaluate the integral along the particular path.

(i) C_1: $z = re^{i0}$ $(0 \le r \le 1)$. Then

$$\int_{C_1} f(z)dz = \int_0^1 \sqrt{r} \cdot 1 dr = \left[\frac{2}{3}r^{3/2}\right]_0^1 = \frac{2}{3}.$$

(ii) C_2: $z = 1 \cdot e^{i\theta}$ $(0 \le \theta \le \pi)$. Then

$$\int_{C_2} f(z)dz = \int_0^\pi e^{i\theta/2} \cdot ie^{i\theta}d\theta = i\int_0^\pi e^{i3\theta/2}d\theta = i\left[\frac{2}{3i}e^{i3\theta/2}\right]_0^\pi = \frac{2}{3}(-i-1) = -\frac{2}{3}(1+i).$$

(*iii*) $-C_3$: $z = re^{i\pi}$ $(0 \le r \le 1)$. Then

$$\int_{C_3} f(z)dz = -\int_{-C_3} f(z)dz = -\int_0^1 \sqrt{r}e^{i\pi/2}(-1)dr = i\int_0^1 \sqrt{r}dr = i\left[\frac{2}{3}r^{3/2}\right]_0^1 = \frac{2}{3}i.$$

The desired result is

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz = \frac{2}{3} - \frac{2}{3}(1+i) + \frac{2}{3}i = 0.$$

The Cauchy-Goursat theorem does not apply since $f(z)$ is not analytic at the origin, or even defined on the negative imaginary axis.

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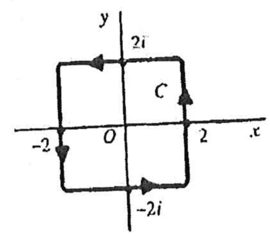
p. 170

1 In this problem, we let C denote the square contour shown in the figure below.



(a) $\displaystyle\int_C \frac{e^{-z}\,dz}{z-(\pi i/2)} = 2\pi i\left[e^{-z}\right]_{z=\pi i/2} = 2\pi i(-i) = 2\pi.$

(b) $\displaystyle\int_C \frac{\cos z}{z(z^2+8)}dz = \int_C \frac{(\cos z)/(z^2+8)}{z-0}dz = 2\pi i\left[\frac{\cos z}{z^2+8}\right]_{z=0} = 2\pi i\left(\frac{1}{8}\right) = \frac{\pi i}{4}.$

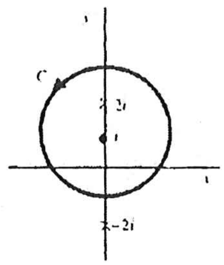
(c) $\displaystyle\int_C \frac{z\,dz}{2z+1} = \int_C \frac{z/2}{z-(-1/2)}dz = 2\pi i\left[\frac{z}{2}\right]_{z=-1/2} = 2\pi i\left(-\frac{1}{4}\right) = -\frac{\pi i}{2}.$

(d) $\displaystyle\int_C \frac{\cosh z}{z^4}dz = \int_C \frac{\cosh z}{(z-0)^{3+1}}dz = \frac{2\pi i}{3!}\left[\frac{d^3}{dz^3}\cosh z\right]_{z=0} = \frac{\pi i}{3}(0) = 0.$

(e) $\displaystyle\int_C \frac{\tan(z/2)}{(z-x_0)^2}dz = \int_C \frac{\tan(z/2)}{(z-x_0)^{1+1}}dz = \frac{2\pi i}{1!}\left[\frac{d}{dz}\tan\left(\frac{z}{2}\right)\right]_{z=x_0}$

$\displaystyle = 2\pi i\left(\frac{1}{2}\sec^2\frac{x_0}{2}\right) = i\pi\sec^2\left(\frac{x_0}{2}\right)$ when $-2 < x_0 < 2.$

2. Let C denote the positively oriented circle $|z-i|=2$, shown below.



(a) The Cauchy integral formula enables us to write

$$\int_C \frac{dz}{z^2+4} = \int_C \frac{dz}{(z-2i)(z+2i)} = \int_C \frac{1/(z+2i)}{z-2i}\,dz = 2\pi i\left(\frac{1}{z+2i}\right)_{z=2i} = 2\pi i\left(\frac{1}{4i}\right) = \frac{\pi}{2}.$$

(4)

$$g(z) = \int_C \frac{s^3+2s}{(s-z)^3}\,ds$$

If z is inside C

$\Rightarrow$

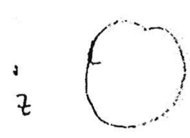
$$g(z) = \int_C \frac{s^3+2s\,ds}{(s-z)^3}$$

$$= -\int_{C_1} \frac{s^3+2s\,ds}{(s-z)^3} = -\frac{2\pi i}{2!}\,\frac{(z^3+2z)'''}{-1} = \frac{-\pi i\,6}{-1}$$

$$= 6\pi i$$

If z is outside of C

$\Rightarrow$



$$g(z) = \int_C \frac{s^3+2s\,ds}{(s-z)^3} = 0$$

5. Suppose that a function f is analytic inside and on a simple closed contour C and that z_0 is not on C. If z_0 is inside C, then

$$\int_C \frac{f'(z)dz}{z-z_0} = 2\pi i f'(z_0) \quad \text{and} \quad \int_C \frac{f(z)dz}{(z-z_0)^2} = \int_C \frac{f(z)dz}{(z-z_0)^{1+1}} = \frac{2\pi i}{1!} f'(z_0).$$

Thus

$$\int_C \frac{f'(z)dz}{z-z_0} = \int_C \frac{f(z)dz}{(z-z_0)^2}.$$

The Cauchy-Goursat theorem tells us that this last equation is also valid when z_0 is exterior to C, each side of the equation being 0.

7. Let C be the unit circle $z = e^{i\theta}$ $(-\pi \le \theta \le \pi)$, and let a denote any real constant. The Cauchy integral formula reveals that

$$\int_C \frac{e^{az}}{z} dz = \int_C \frac{e^{az}}{z-0} dz = 2\pi i \left[e^{az}\right]_{z=0} = 2\pi i.$$

On the other hand, the stated parametric representation for C gives us

$$\int_C \frac{e^{az}}{z} dz = \int_{-\pi}^{\pi} \frac{\exp(ae^{i\theta})}{e^{i\theta}} ie^{i\theta} d\theta = i\int_{-\pi}^{\pi} \exp[a(\cos\theta + i\sin\theta)]d\theta$$

$$= i\int_{-\pi}^{\pi} e^{a\cos\theta} e^{ia\sin\theta} d\theta = i\int_{-\pi}^{\pi} e^{a\cos\theta}[\cos(a\sin\theta) + i\sin(a\sin\theta)]d\theta$$

$$= -\int_{-\pi}^{\pi} e^{a\cos\theta}\sin(a\sin\theta)d\theta + i\int_{-\pi}^{\pi} e^{a\cos\theta}\cos(a\sin\theta)d\theta.$$

Equating these two different expressions for the integral $\int_C \frac{e^{az}}{z} dz$, we have

$$-\int_{-\pi}^{\pi} e^{a\cos\theta}\sin(a\sin\theta)d\theta + i\int_{-\pi}^{\pi} e^{a\cos\theta}\cos(a\sin\theta)d\theta = 2\pi i.$$

Then, by equating the imaginary parts on each side of this last equation, we see that

$$\int_{-\pi}^{\pi} e^{a\cos\theta}\cos(a\sin\theta)d\theta = 2\pi;$$

and, since the integrand here is even,

$$\int_{0}^{\pi} e^{a\cos\theta}\cos(a\sin\theta)d\theta = \pi.$$

8. (a) The binomial formula enables us to write

(optional)

$$P_n(z)=\frac{1}{n!2^n}\frac{d^n}{dz^n}(z^2-1)^n=\frac{1}{n!2^n}\frac{d^n}{dz^n}\sum_{k=0}^{n}\binom{n}{k}z^{2n-2k}(-1)^k.$$

We note that the highest power of z appearing under the derivative is z^{2n}, and differentiating it n times brings it down to z^n. So $P_n(z)$ is a polynomial of degree n.

(b) We let C denote any positively oriented simple closed contour surrounding a fixed point z. The Cauchy integral formula for derivatives tells us that

$$\frac{d^n}{dz^n}(z^2-1)^n=\frac{n!}{2\pi i}\int_C\frac{(s^2-1)^n}{(s-z)^{n+1}}ds \qquad (n=0,1,2,\ldots).$$

Hence the polynomials $P_n(z)$ in part (a) can be written

$$P_n(z)=\frac{1}{2^{n+1}\pi i}\int_C\frac{(s^2-1)^n}{(s-z)^{n+1}}ds \qquad (n=0,1,2,\ldots).$$

(c) Note that

$$\frac{(s^2-1)^n}{(s-1)^{n+1}}=\frac{(s-1)^n(s+1)^n}{(s-1)^{n+1}}=\frac{(s+1)^n}{s-1}.$$

Referring to the final result in part (b), then, we have

$$P_n(1)=\frac{1}{2^{n+1}\pi i}\int_C\frac{(s^2-1)^n}{(s-1)^{n+1}}ds=\frac{1}{2^n}\frac{1}{2\pi i}\int_C\frac{(s+1)^n}{s-1}ds=\frac{1}{2^n}2^n=1 \qquad (n=0,1,2,\ldots).$$

Also, since

$$\frac{(s^2-1)^n}{(s+1)^{n+1}}=\frac{(s-1)^n(s+1)^n}{(s+1)^{n+1}}=\frac{(s-1)^n}{s+1},$$

we have

$$P_n(-1)=\frac{1}{2^{n+1}\pi i}\int_C\frac{(s^2-1)^n}{(s+1)^{n+1}}ds=\frac{1}{2^n}\frac{1}{2\pi i}\int_C\frac{(s-1)^n}{s+1}ds=\frac{1}{2^n}(-2)^n=(-1)^n \qquad (n=0,1,2,\ldots).$$

(9.) We are asked to show that

(optional)

$$f''(z) = \frac{1}{\pi i}\int_C \frac{f(s)ds}{(s-z)^3}.$$

(a) In view of the expression for $f'(z)$ in the lemma,

$$\frac{f'(z+\Delta z) - f'(z)}{\Delta z} = \frac{1}{2\pi i}\int_C \left[\frac{1}{(s-z-\Delta z)^2} - \frac{1}{(s-z)^2}\right]\frac{f(s)ds}{\Delta z}$$
$$= \frac{1}{2\pi i}\int_C \frac{2(s-z)-\Delta z}{(s-z-\Delta z)^2(s-z)^2} f(s)ds.$$

Then

$$\frac{f'(z+\Delta z) - f'(z)}{\Delta z} - \frac{1}{\pi i}\int_C \frac{f(s)ds}{(s-z)^3} = \frac{1}{2\pi i}\int_C \left[\frac{2(s-z)-\Delta z}{(s-z-\Delta z)^2(s-z)^2} - \frac{2}{(s-z)^3}\right] f(s)ds$$
$$= \frac{1}{2\pi i}\int_C \frac{3(s-z)\Delta z - 2(\Delta z)^2}{(s-z-\Delta z)^2(s-z)^3} f(s)ds.$$

(b) We must show that

$$\left|\int_C \frac{3(s-z)\Delta z - 2(\Delta z)^2}{(s-z-\Delta z)^2(s-z)^3} f(s)ds\right| \le \frac{(3D|\Delta z| + 2|\Delta z|^2)M}{(d-|\Delta z|)^2 d^3} L.$$

Now D, d, M, and L are as in the statement of the exercise in the text. The triangle inequality tells us that

$$|3(s-z)\Delta z - 2(\Delta z)^2| \le 3|s-z||\Delta z| + 2|\Delta z|^2 \le 3D|\Delta z| + 2|\Delta z|^2.$$

Also, we know from the verification of the expression for $f'(z)$ in the lemma that $|s-z-\Delta z| \ge d - |\Delta z| > 0$; and this means that

$$|(s-z-\Delta z)^2(s-z)^3| \ge (d-|\Delta z|)^2 d^3 > 0.$$

This gives the desired inequality.

(c) If we let Δz tend to 0 in the inequality obtained in part *(b)* we find that

$$\lim_{\Delta z \to 0} \frac{1}{2\pi i}\int_C \frac{3(s-z)\Delta z - 2(\Delta z)^2}{(s-z-\Delta z)^2(s-z)^3} f(s)ds = 0.$$

This, together with the result in part *(a)*, yields the desided expression for $f''(z)$.

p. 178.

(1) $u \le u_0 \ \forall (x,y)$

$f = u + iv$ - entire function

Define $g(z) = e^{f(z)}$

$\Rightarrow |g(z)| = |e^{u+iv}| = e^u \le e^{u_0}$

$\Rightarrow$ $g(z)$ is entire and bounded

$\Rightarrow$ by ~~Co.~~ Liouville's thm

$g(z) = \text{const} \Rightarrow f(z) = \text{Const}.$.

(2) $w = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \dots \frac{a_{n-1}}{z}.$

Choose R so large that $\forall_{0 \le j \le n-1}: \left|\frac{a_j}{R^{n-j}}\right| < \frac{|a_n|}{n}$

$\Rightarrow$ for $|z| > R$: $\left|\frac{a_j}{z^{n-j}}\right| \le \left|\frac{a_j}{R^{n-j}}\right| < \frac{|a_n|}{n}$

$\Rightarrow |P(z)| = |a_n + w| \, |z|^n \le \left(|a_n| + \left|\frac{a_0}{z_n}\right| + \dots + \left|\frac{a_{n-1}}{z}\right|\right)|z|^n$

$< \left(|a_n| + n \cdot \frac{|a_n|}{n}\right)|z|^n = 2|a_n||z|^n$ ⊠.