math 561 HW 06 Solutions

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1. Let us use definition (2), Sec. 55, to show that the sequence

$$z_n = -2 + i\frac{(-1)^n}{n^2} \qquad (n = 1, 2, \ldots)$$

converges to -2. Observe that $|z_n - (-2)| = \frac{1}{n^2}$. Thus, for each $\varepsilon > 0$,

$$|z_n - (-2)| < \varepsilon \quad \text{whenever} \quad n > n_0,$$

where n_0 is any positive integer such that $n_0 \geq \frac{1}{\sqrt{\varepsilon}}$.

2. Note that if $z_n = 2 + i\frac{(-1)^n}{n^2}$ $(n = 1, 2, \ldots)$, then

$$\Theta_{2n} = \operatorname{Arg} z_{2n} \to 0 \quad \text{and} \quad \Theta_{2n-1} = \operatorname{Arg} z_{2n-1} \to 0 \qquad (n = 1, 2, \ldots)$$

Hence the sequence Θ_n $(n = 1, 2, \ldots)$ does converge.

3. Suppose that $\lim_{n\to\infty} z_n = z$. That is, for each $\varepsilon > 0$, there is a positive integer n_0 such that $|z_n - z| < \varepsilon$ whenever $n > n_0$. In view of the inequality (see Sec. 4)

$$|z_n - z| \geq ||z_n| - |z||,$$

it follows that $||z_n| - |z|| < \varepsilon$ whenever $n > n_0$. That is, $\lim_{n\to\infty} |z_n| = |z|$.

4. The summation formula found in the example in Sec. 56 can be written

$$\sum_{n=1}^{\infty} z^n = \frac{z}{1-z} \quad \text{when} \quad |z| < 1.$$

If we put $z = re^{i\theta}$, where $0 < r < 1$, the left-hand side becomes

$$\sum_{n=1}^{\infty}(re^{i\theta})^n = \sum_{n=1}^{\infty} r^n e^{in\theta} = \sum_{n=1}^{\infty} r^n \cos n\theta + i\sum_{n=1}^{\infty} r^n \sin n\theta;$$

and the right-hand side takes the form

$$\frac{re^{i\theta}}{1-re^{i\theta}} \cdot \frac{1-re^{-i\theta}}{1-re^{-i\theta}} = \frac{re^{i\theta}-r^2}{1-r(e^{i\theta}+e^{-i\theta})+r^2} = \frac{r\cos\theta - r^2 + ir\sin\theta}{1-2r\cos\theta + r^2}.$$

Thus

$$\sum_{n=1}^{\infty} r^n \cos n\theta + i\sum_{n=1}^{\infty} r^n \sin n\theta = \frac{r\cos\theta - r^2}{1-2r\cos\theta + r^2} + i\frac{r\sin\theta}{1-2r\cos\theta+r^2}.$$

Equating the real parts on each side here and then the imaginary parts, we arrive at the summation formulas

$$\sum_{n=1}^{\infty} r^n \cos n\theta = \frac{r\cos\theta - r^2}{1-2r\cos\theta + r^2} \quad \text{and} \quad \sum_{n=1}^{\infty} r^n \sin n\theta = \frac{r\sin\theta}{1-2r\cos\theta+r^2},$$

where $0 < r < 1$. These formulas clearly hold when $r = 0$ too.

6. Suppose that $\sum_{n=1}^{\infty} z_n = S$. To show that $\sum_{n=1}^{\infty} \overline{z}_n = \overline{S}$, we write $z_n = x_n + iy_n$, $S = X + iY$ and appeal to the theorem in Sec. 56. First of all, we note that

$$\sum_{n=1}^{\infty} x_n = X \quad \text{and} \quad \sum_{n=1}^{\infty} y_n = Y.$$

Then, since $\sum_{n=1}^{\infty}(-y_n) = -Y$, it follows that

$$\sum_{n=1}^{\infty} \overline{z}_n = \sum_{n=1}^{\infty}(x_n - iy_n) = \sum_{n=1}^{\infty}[x_n + i(-y_n)] = X - iY = \overline{S}.$$

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$z_n \to z$ for $n \to \infty$

choose n_0 such that $|z_n - z| < \varepsilon$ for $n > n_0$

a) $\Rightarrow |z_n| = |z + (z_n - z)| < |z| + 1 = M_0 \quad \forall n > n_0$

choose $M = \max(M_0, |z_1|, \dots |z_{n-1}|)$

b) $z_n = x_n + i y_n$.

From convergence of x_n, y_n

$\Rightarrow |x_n| \le M_1, |y_n| \le y_2$

$\Rightarrow |z_n| = \sqrt{x_n^2 + y_n^2} \le \sqrt{M_1^2 + M_2^2} = M$.

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1. Replace z by z^2 in the known series

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \qquad (|z| < \infty)$$

to get

$$\cosh(z^2) = \sum_{n=0}^{\infty} \frac{z^{4n}}{(2n)!} \qquad (|z| < \infty).$$

Then, multiplying through this last equation by z, we have the desired result:

$$z\cosh(z^2) = \sum_{n=0}^{\infty} \frac{z^{4n+1}}{(2n)!} \qquad (|z| < \infty).$$

2. *(b)* Replacing z by $z-1$ in the known expansion

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad (|z| < \infty),$$

we have

$$e^{z-1} = \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} \qquad (|z| < \infty).$$

So

$$e^z = e^{z-1}e = e\sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} \qquad (|z| < \infty).$$

3. We want to find the Maclaurin series for the function

$$f(z)=\frac{z}{z^4+9}=\frac{z}{9}\cdot\frac{1}{1+(z^4/9)}.$$

To do this, we first replace z by $-(z^4/9)$ in the known expansion

$$\frac{1}{1-z}=\sum_{n=0}^{\infty} z^n \qquad (|z|<1),$$

as well as its condition of validity, to get

$$\frac{1}{1+(z^4/9)}=\sum_{n=0}^{\infty}\frac{(-1)^n}{3^{2n}}z^{4n} \qquad (|z|<\sqrt{3}).$$

Then, if we multiply through this last equation by $\frac{z}{9}$, we have the desired expansion:

$$f(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{3^{2n+2}}z^{4n+1} \qquad (|z|<\sqrt{3}).$$

4 $f(z) = \sin z.$

(1) $n=0 \Rightarrow f^{(0)}(0) = \sin 0 = 0,$

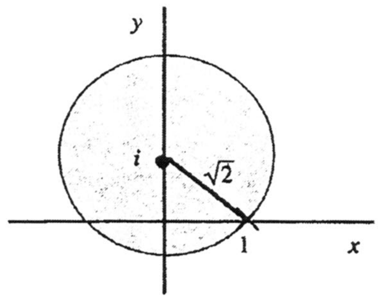
$f'(0) = \cos 0 = 1 = (-1)^0$

(2) ~~$n \to 0$~~ $n > 0$

$(\sin z)^{(2n)}\Big|_{z=0} (-1)^n \sin 0 = 0$

$(\sin z)^{2n+1} = (-1)^n \cos 0 = (-1)^n$

7.) The function $\frac{1}{1-z}$ has a singularity at $z=1$. So the Taylor series about $z=i$ is valid when $|z-i|<\sqrt{2}$, as indicated in the figure below.



To find the series, we start by writing

$$\frac{1}{1-z}=\frac{1}{(1-i)-(z-i)}=\frac{1}{1-i}\cdot\frac{1}{1-(z-i)/(1-i)}.$$

This suggests that we replace z by $(z-i)/(1-i)$ in the known expansion

$$\frac{1}{1-z}=\sum_{n=0}^{\infty}z^n \qquad (|z|<1)$$

and then multiply through by $\frac{1}{1-i}$. The desired Taylor series is then obtained:

$$\frac{1}{1-z}=\sum_{n=0}^{\infty}\frac{(z-i)^n}{(1-i)^{n+1}} \qquad (|z-i|<\sqrt{2}).$$

9.) The identity $\sinh(z+\pi i)=-\sinh z$ and the periodicity of $\sinh z$, with period $2\pi i$, tell us that

$$\sinh z=-\sinh(z+\pi i)=-\sinh(z-\pi i).$$

So, if we replace z by $z-\pi i$ in the known representation

$$\sinh z=\sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+1)!} \qquad (|z|<\infty)$$

and then multiply through by -1, we find that

$$\sinh z=-\sum_{n=0}^{\infty}\frac{(z-\pi i)^{2n+1}}{(2n+1)!} \qquad (|z-\pi i|<\infty).$$

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(a) $$\frac{\sinh z}{z^2} = \frac{\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}}{z^2}$$

$$= \frac{z}{z^2} + \frac{\sum_{n=1}^{\infty} \frac{z^{2n+1}}{(2n+1)!}}{z^2}$$

$$= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z^{2n-1}}{(2n+1)!} = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+3)!}, \quad (0 < |z| < \infty)$$

$n \to n+1$

(b)

$$z^3 \cosh\left(\frac{1}{z}\right) = z^3 \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^{2n} \frac{1}{(2n)!}$$

$$= z^3 + z^3 \left(\frac{1}{z}\right)^2 \frac{1}{2} + z^3 \sum_{n=2}^{\infty} \left(\frac{1}{z}\right)^{2n} \frac{1}{(2n)!}$$

$n \to n+1$

$$= z^3 + \frac{z}{2} + \sum_{n=1}^{\infty} \frac{1}{z^{2n-1}(2n+2)!} \qquad (0 < |z| < \infty)$$

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1. We may use the expansion

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \qquad (|z| < \infty)$$

to see that when $0 < |z| < \infty$,

$$z^2 \sin\left(\frac{1}{z^2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot \frac{1}{z^{4n}} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot \frac{1}{z^{4n}}.$$

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$$\frac{e^{z+1-1}}{(z+1)^2} = \frac{e^{-1}}{(z+1)^2} \sum_{n=0}^{\infty} \frac{(z+1)^n}{n!}$$

$$= e^{-1} \left[\frac{1}{(z+1)^2} + \frac{1}{z+1} + \sum_{n=2}^{\infty} \frac{(z+1)^{n-2}}{n!} \right]$$

$$= e^{-1} \left[\frac{1}{(z+1)^2} + \frac{1}{z+1} + \sum_{n=0}^{\infty} \frac{(z+1)^n}{(n+2)!} \right] \qquad 0 < |z+1| < \infty$$

3. Suppose that $1 < |z| < \infty$ and recall the Maclaurin series representation

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad (|z| < 1).$$

This enables us to write

$$\frac{1}{1+z} = \frac{1}{z} \cdot \frac{1}{1 + \frac{1}{z}} = \frac{1}{z} \sum_{n=0}^{\infty} \left(-\frac{1}{z}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{n+1}} \qquad (1 < |z| < \infty).$$

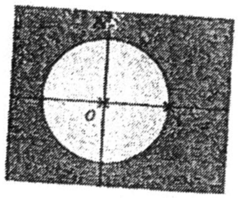
Replacing n by $n-1$ in this last series and then noting that

$$(-1)^{n-1} = (-1)^{n-1}(-1)^2 = (-1)^{n+1},$$

we arrive at the desired expansion:

$$\frac{1}{1+z} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{z^n} \qquad (1 < |z| < \infty).$$

4. The singularities of the function $f(z)=\dfrac{1}{z^2(1-z)}$ are at the points $z=0$ and $z=1$. Hence there are Laurent series in powers of z for the domains $0<|z|<1$ and $1<|z|<\infty$ (see the figure below).



To find the series when $0<|z|<1$, recall that $\dfrac{1}{1-z}=\sum_{n=0}^{\infty}z^n$ $(|z|<1)$ and write

$$f(z)=\frac{1}{z^2}\cdot\frac{1}{1-z}=\frac{1}{z^2}\sum_{n=0}^{\infty}z^n=\sum_{n=0}^{\infty}z^{n-2}=\frac{1}{z^2}+\frac{1}{z}+\sum_{n=2}^{\infty}z^{n-2}=\sum_{n=0}^{\infty}z^n+\frac{1}{z}+\frac{1}{z^2}.$$

As for the domain $1<|z|<\infty$, note that $|1/z|<1$ and write

$$f(z)=-\frac{1}{z^3}\cdot\frac{1}{1-(1/z)}=-\frac{1}{z^3}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n=-\sum_{n=0}^{\infty}\frac{1}{z^{n+3}}=-\sum_{n=3}^{\infty}\frac{1}{z^n}.$$

6. $0<|z-1|<2$

$$\frac{z}{(z-1)(z-3)}=\frac{z-3+3}{(z-1)(z-3)}=\frac{1}{z-1}+\frac{3}{(z-1)(-2+z-1)}$$

$$=\frac{1}{z-1}+\frac{3}{(z-1)(-2)}\frac{1}{\left(1-\frac{z-1}{2}\right)}=\frac{1}{z-1}+\frac{3}{(-2)(z-1)}\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^n}$$

$$=\frac{1}{z-1}+\frac{3}{(-2)(z-1)}+\frac{3}{(-2)}\sum_{n=1}^{\infty}\frac{(z-1)^{n-1}}{2^n}$$

$$=-\frac{1}{z-1}-3\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}}$$

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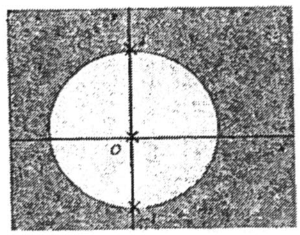
7. The function $f(z)=\dfrac{1}{z(1+z^2)}$ has isolated singularities at $z=0$ and $z=\pm i$, as indicated in the figure below. Hence there is a Laurent series representation for the domain $0<|z|<1$ and also one for the domain $1<|z|<\infty$, which is exterior to the circle $|z|=1$.



To find each of these Laurent series, we recall the Maclaurin series representation

$$\frac{1}{1-z}=\sum_{n=0}^{\infty} z^n \qquad (|z|<1).$$

For the domain $0<|z|<1$, we have

$$f(z)=\frac{1}{z}\cdot\frac{1}{1+z^2}=\frac{1}{z}\sum_{n=0}^{\infty}(-z^2)^n=\sum_{n=0}^{\infty}(-1)^n z^{2n-1}=\frac{1}{z}+\sum_{n=1}^{\infty}(-1)^n z^{2n-1}=\sum_{n=0}^{\infty}(-1)^{n+1}z^{2n+1}+\frac{1}{z}.$$

On the other hand, when $1<|z|<\infty$,

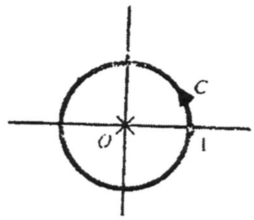
$$f(z)=\frac{1}{z^3}\cdot\frac{1}{1+\frac{1}{z^2}}=\frac{1}{z^3}\sum_{n=0}^{\infty}\left(-\frac{1}{z^2}\right)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n+3}}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{z^{2n+1}}.$$

In this second expansion, we have used the fact that $(-1)^{n-1}=(-1)^{n-1}(-1)^2=(-1)^{n+1}$.

10. (a) Let z be any fixed complex number and C the unit circle $w=e^{i\phi}$ $(-\pi\le\phi\le\pi)$ in the w plane. The function

$$f(w)=\exp\left[\frac{z}{2}\left(w-\frac{1}{w}\right)\right]$$

has the one singularity $w=0$ in the w plane. That singularity is, of course, interior to C, as shown in the figure below.



Now the function $f(w)$ has a Laurent series representation in the domain $0<|w|<\infty$. According to expression (5), Sec. 55, then,

$$\exp\left[\frac{z}{2}\left(w-\frac{1}{w}\right)\right]=\sum_{n=-\infty}^{\infty}J_n(z)w^n \qquad (0<|w|<\infty),$$

where the coefficients $J_n(z)$ are

$$J_n(z)=\frac{1}{2\pi i}\int_C \frac{\exp\left[\frac{z}{2}\left(w-\frac{1}{w}\right)\right]}{w^{n+1}}dw \qquad (n=0,\pm1,\pm2,\ldots).$$

Using the parametric representation $w=e^{i\phi}$ $(-\pi\le\phi\le\pi)$ for C, let us rewrite this expression for $J_n(z)$ as follows:

$$J_n(z)=\frac{1}{2\pi i}\int_{-\pi}^{\pi}\frac{\exp\left[\frac{z}{2}\left(e^{i\phi}-e^{-i\phi}\right)\right]}{e^{i(n+1)\phi}}ie^{i\phi}d\phi=\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp[iz\sin\phi]e^{-in\phi}d\phi.$$

That is,

$$J_n(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp[-i(n\phi-z\sin\phi)]d\phi \qquad (n=0,\pm1,\pm2,\ldots).$$

(b) The last expression for $J_n(z)$ in part *(a)* can be written as

$$\begin{aligned}J_n(z)&=\frac{1}{2\pi}\int_{-\pi}^{\pi}[\cos(n\phi-z\sin\phi)-i\sin(n\phi-z\sin\phi)]d\phi\\&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(n\phi-z\sin\phi)d\phi-\frac{i}{2\pi}\int_{-\pi}^{\pi}\sin(n\phi-z\sin\phi)d\phi\\&=\frac{1}{2\pi}2\int_{0}^{\pi}\cos(n\phi-z\sin\phi)d\phi-\frac{i}{2\pi}0 \qquad (n=0,\pm1,\pm2,\ldots).\end{aligned}$$

That is,

$$J_n(z)=\frac{1}{\pi}\int_{0}^{\pi}\cos(n\phi-z\sin\phi)d\phi \qquad (n=0,\pm1,\pm2,\ldots).$$

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1. Differentiating each side of the representation

$$\frac{1}{1-z}=\sum_{n=0}^{\infty} z^n \qquad (|z|<1),$$

we find that

$$\frac{1}{(1-z)^2}=\frac{d}{dz}\sum_{n=0}^{\infty} z^n=\sum_{n=0}^{\infty}\frac{d}{dz}z^n=\sum_{n=1}^{\infty} nz^{n-1}=\sum_{n=0}^{\infty}(n+1)z^n \qquad (|z|<1).$$

Another differentiation gives

$$\frac{2}{(1-z)^3}=\frac{d}{dz}\sum_{n=0}^{\infty}(n+1)z^n=\sum_{n=0}^{\infty}(n+1)\frac{d}{dz}z^n=\sum_{n=1}^{\infty}n(n+1)z^{n-1}=\sum_{n=0}^{\infty}(n+1)(n+2)z^n \qquad (|z|<1).$$

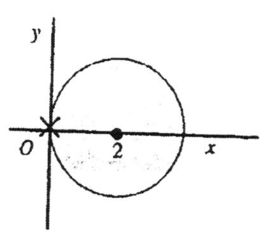
2. Replace z by $1/(1-z)$ on each side of the Maclaurin series representation (Exercise 1)

$$\frac{1}{(1-z)^2}=\sum_{n=0}^{\infty}(n+1)z^n \qquad (|z|<1),$$

as well as in its condition of validity. This yields the Laurent series representation

$$\frac{1}{z^2}=\sum_{n=2}^{\infty}\frac{(-1)^n(n-1)}{(z-1)^n} \qquad (1<|z-1|<\infty).$$

3. Since the function $f(z)=1/z$ has a singular point at $z=0$, its Taylor series about $z_0=2$ is valid in the open disk $|z-2|<2$, as indicated in the figure below.



To find that series, write

$$\frac{1}{z}=\frac{1}{2+(z-2)}=\frac{1}{2}\cdot\frac{1}{1+(z-2)/2}$$

to see that it can be obtained by replacing z by $-(z-2)/2$ in the known expansion

Specifically,

$$\frac{1}{1-z}=\sum_{n=0}^{\infty} z^n \qquad (|z|<1).$$

or

$$\frac{1}{z}=\frac{1}{2}\sum_{n=0}^{\infty}\left[-\frac{(z-2)}{2}\right]^n \qquad (|z-2|<2),$$

$$\frac{1}{z}=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(z-2)^n \qquad (|z-2|<2).$$

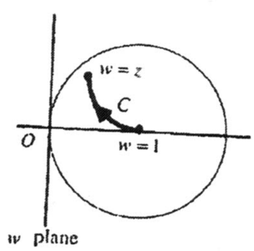
Differentiating this series term by term, we have

$$-\frac{1}{z^2}=\sum_{n=1}^{\infty}\frac{(-1)^n}{2^{n+1}}n(z-2)^{n-1}=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2^{n+2}}(n+1)(z-2)^n \qquad (|z-2|<2).$$

Thus

$$\frac{1}{z^2}=\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n(n+1)\left(\frac{z-2}{2}\right)^n \qquad (|z-2|<2).$$

6. Let C be a contour lying in the open disk $|w-1|<1$ in the w plane that extends from the point $w=1$ to a point $w=z$, as shown in the figure below.



According to Theorem 1 in Sec. 65, we can integrate the Taylor series representation

$$\frac{1}{w}=\sum_{n=0}^{\infty}(-1)^n(w-1)^n \qquad (|w-1|<1)$$

term by term along the contour C. Thus

$$\int_C\frac{dw}{w}=\int_C\sum_{n=0}^{\infty}(-1)^n(w-1)^n\,dw=\sum_{n=0}^{\infty}(-1)^n\int_C(w-1)^n\,dw.$$

But

$$\int_C\frac{dw}{w}=\int_1^z\frac{dw}{w}=[\operatorname{Log} w]_1^z=\operatorname{Log} z-\operatorname{Log} 1=\operatorname{Log} z$$

and

$$\int_C(w-1)^n=\int_1^z(w-1)^n\,dw=\left[\frac{(w-1)^{n+1}}{n+1}\right]_1^z=\frac{(z-1)^{n+1}}{n+1}.$$

Hence

$$\operatorname{Log} z = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}(z-1)^{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(z-1)^n \qquad (|z-1|<1);$$

and, since $(-1)^{n-1} = (-1)^{n-1}(-1)^2 = (-1)^{n+1}$, this result becomes

$$\operatorname{Log} z = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(z-1)^n \qquad (|z-1|<1).$$

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$$f_2(z) = \frac{1}{z^2} \quad (z \neq 0)$$

$$f_2(z) = \frac{1}{(z+1-1)^2} = \frac{1}{(1-(z+1))^2} = \frac{d}{dz}\frac{1}{1-(1+z)}$$

$$= \frac{d}{dz}\sum_{n=0}^{\infty}(1+z)^n = \sum_{n=1}^{\infty} n(1+z)^n$$

$$= \sum_{n=0}^{\infty}(n+1)(1+z)^{n+1} \quad (|z+1|<1)$$

$\Rightarrow$ $f_1(z) = f_2(z)$ for $(|z+1|<1)$

$\Rightarrow$ from analytic continuation $f_1(z) = f_2(z)$ except $z=0$.

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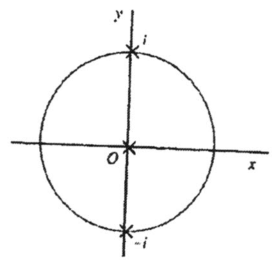
1. The singularities of the function $f(z)=\dfrac{e^z}{z(z^2+1)}$ are at $z=0,\pm i$. The problem here is to find the Laurent series for f that is valid in the punctured disk $0<|z|<1$, shown below.



We begin by recalling the Maclaurin series representations

$$e^z = 1+\frac{z}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots \qquad (|z|<\infty)$$

and

$$\frac{1}{1-z}=1+z+z^2+z^3+\cdots \qquad (|z|<1),$$

which enable us to write

$$e^z = 1+z+\frac{1}{2}z^2+\frac{1}{6}z^3+\cdots \qquad (|z|<\infty)$$

and

$$\frac{1}{z^2+1}=1-z^2+z^4-z^6+\cdots \qquad (|z|<1).$$

Multiplying these last two series term by term, we have the Maclaurin series representation

$$\begin{aligned}\frac{e^z}{z^2+1} &= 1+z+\frac{1}{2}z^2+\frac{1}{6}z^3+\cdots \\ &\qquad\quad -z^2 \quad -z^3-\cdots \\ &\qquad\qquad\qquad\qquad z^4+\cdots \\ &\qquad\qquad\qquad\qquad\quad \vdots \\ &= 1+z-\frac{1}{2}z^2-\frac{5}{6}z^3+\cdots,\end{aligned}$$

which is valid when $|z|<1$. The desired Laurent series is then obtained by multiplying each side of the above representation by $\dfrac{1}{z}$:

$$\frac{e^z}{z(z^2+1)}=\frac{1}{z}+1-\frac{1}{2}z-\frac{5}{6}z^2+\cdots \qquad (0<|z|<1).$$

(2)

$$\csc z = \frac{1}{\sin z} = \frac{1}{z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdot}$$

$$= \frac{1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots - \left(-\frac{z^2}{3!} + \frac{z^4}{5!} + \cdots\right)}{z\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!}\right)}$$

$$= \frac{1}{z} + \frac{\frac{z^2}{3!}}{z} \frac{\left[1 - \frac{z^2 3!}{5!} + \cdots\right]}{\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdot\right)}$$

$$= \frac{1}{z} + \frac{z}{3!} \frac{\left[\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots\right) + \frac{z^2}{3!} - \frac{z^4}{5!} - \frac{z^2 3!}{5!} + \cdots\right]}{1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdot}$$

$$= \frac{1}{z} + \frac{z}{3!} + \frac{z \cdot z^2}{3!}\left(\frac{1}{3!} - \frac{3!}{5!}\right) + \cdots ?$$

$$= \frac{1}{z} + \frac{z}{3!} + z^3\left[\frac{1}{(3!)^2} - \frac{1}{5!}\right] + \cdots$$

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(a) $$\frac{1}{1+z^2/3!+z^4/5!+\cdots} = d_0 + d_1 z + d_2 z^2 + \cdots$$

$$\Rightarrow 1 = \left(1+\frac{z^2}{3!}+\frac{z^4}{5!}+\cdots\right)\left(d_0+d_1 z+d_2 z^2+d_3 z^3+d_4 z^4+\cdots\right)$$

$$= d_0 + d_1 z + z^2\left(d_2+\frac{1}{3!}d_0\right) + \left(d_3+\frac{d_1}{3!}\right)z^3 + \left(d_4+\frac{d_2}{3!}+\frac{d_0}{5!}\right)z^4+\cdots = 0$$

$|z| < \pi$

$$\Rightarrow \begin{array}{c|l} z^0 & d_0 = 1 \\ z^1 & d_1 = 0 \\ z^2 & d_2 = -\frac{1}{3!}d_0 = -\frac{1}{3!} = -\frac{1}{6} \\ z^3 & d_3 + 0 = 0 \Rightarrow d_3 = 0 \\ z^4 & d_4 + \frac{d_2}{3!} + \frac{d_0}{5!} = d_4 - \frac{1}{(3!)^2} + \frac{1}{5!} = 0 \end{array}$$

$$d_4 = \frac{1}{(3!)^2} - \frac{1}{5!} = \frac{7}{360}$$

$\Rightarrow$