math 561 HW 07 solutions

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1. (a) Let us write

$$\frac{1}{z+z^2}=\frac{1}{z}\cdot\frac{1}{1+z}=\frac{1}{z}(1-z+z^2-z^3+\cdots)=\frac{1}{z}-1+z-z^2+\cdots \qquad (0<|z|<1).$$

The residue at $z=0$, which is the coefficient of $\frac{1}{z}$, is clearly 1.

(b) We may use the expansion

$$\cos z = 1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots \qquad (|z|<\infty)$$

to write

$$z\cos\left(\frac{1}{z}\right)=z\left(1-\frac{1}{2!}\cdot\frac{1}{z^2}+\frac{1}{4!}\cdot\frac{1}{z^4}-\frac{1}{6!}\cdot\frac{1}{z^6}+\cdots\right)=z-\frac{1}{2!}\cdot\frac{1}{z}+\frac{1}{4!}\cdot\frac{1}{z^3}-\frac{1}{6!}\cdot\frac{1}{z^5}+\cdots$$

$$(0<|z|<\infty).$$

The residue at $z=0$, or coefficient of $\frac{1}{z}$, is now seen to be $-\frac{1}{2}$.

(c) Observe that

$$\frac{z-\sin z}{z}=\frac{1}{z}(z-\sin z)=\frac{1}{z}\left[z-\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right)\right]=\frac{z^2}{3!}-\frac{z^4}{5!}+\cdots \qquad (0<|z|<\infty).$$

Since the coefficient of $\frac{1}{z}$ in this Laurent series is 0, the residue at $z=0$ is 0.

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2. In each part, C denotes the positively oriented circle $|z| = 3$.

(a) To evaluate $\int_C \frac{\exp(-z)}{z^2}\,dz$, we need the residue of the integrand at $z = 0$. From the Laurent series

$$\frac{\exp(-z)}{z^2} = \frac{1}{z^2}\left(1 - \frac{z}{1!} + \frac{z^2}{2!} - \frac{z^3}{3!} + \cdots\right) = \frac{1}{z^2} - \frac{1}{1!}\cdot\frac{1}{z} + \frac{1}{2!} - \frac{z}{3!} + \cdots \qquad (0 < |z| < \infty),$$

we see that the required residue is -1. Thus

$$\int_C \frac{\exp(-z)}{z^2}\,dz = 2\pi i(-1) = -2\pi i.$$

(b) $|z| = 3$

$$f(z) = \frac{e^{-(z-1)-1}}{(z-1)^2} = \frac{1 - (z-1) + \frac{(z-1)^2}{2!} + \cdots}{(z-1)^2}\, e^{-1} \qquad 0 < |z-1| < \infty$$

$$\Rightarrow \operatorname*{Res}_{z=1} \frac{e^{-z}}{(z-1)^2} = -e^{-1}$$

$$\Rightarrow \int_C f(z)\,dz = -2\pi i\, e^{-1}$$

(c) Likewise, to evaluate the integral $\int_C z^2 \exp\left(\frac{1}{z}\right) dz$, we must find the residue of the integrand at $z=0$. The Laurent series

$$z^2 \exp\left(\frac{1}{z}\right) = z^2\left(1+\frac{1}{1!}\cdot\frac{1}{z}+\frac{1}{2!}\cdot\frac{1}{z^2}+\frac{1}{3!}\cdot\frac{1}{z^3}+\frac{1}{4!}\cdot\frac{1}{z^4}+\cdots\right)$$

$$= z^2+\frac{z}{1!}+\frac{1}{2!}+\frac{1}{3!}\cdot\frac{1}{z}+\frac{1}{4!}\cdot\frac{1}{z^2}+\cdots,$$

which is valid for $0<|z|<\infty$, tells us that the needed residue is $\frac{1}{6}$. Hence

$$\int_C z^2 \exp\left(\frac{1}{z}\right) dz = 2\pi i\left(\frac{1}{6}\right) = \frac{\pi i}{3}.$$

(3.) In each part of this problem, C is the positively oriented circle $|z|=2$.

(a) If $f(z)=\frac{z^5}{1-z^3}$, then

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^7-z^4} = -\frac{1}{z^4}\cdot\frac{1}{1-z^3} = -\frac{1}{z^4}(1+z^3+z^6+\cdots) = -\frac{1}{z^4}-\frac{1}{z}-z^2-\cdots$$

when $0<|z|<1$. This tells us that

$$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right) = 2\pi i(-1) = -2\pi i.$$

4. Let C denote the circle $|z| = 1$, taken counterclockwise.

(a) The Maclaurin series $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ $(|z| < \infty)$ enables us to write

$$\int_C \exp\left(z + \frac{1}{z}\right) dz = \int_C e^z e^{1/z}\, dz = \int_C e^{1/z} \sum_{n=0}^{\infty} \frac{z^n}{n!}\, dz = \sum_{n=0}^{\infty} \frac{1}{n!} \int_C z^n \exp\left(\frac{1}{z}\right) dz.$$

(b) Referring to the Maclaurin series for e^z once again, let us write

$$z^n \exp\left(\frac{1}{z}\right) = z^n \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{1}{z^k} = \sum_{k=0}^{\infty} \frac{1}{k!} z^{n-k} \qquad (n = 0, 1, 2, \ldots).$$

Now the $\frac{1}{z}$ in this series occurs when $n - k = -1$, or $k = n + 1$. So, by the residue theorem,

$$\int_C z^n \exp\left(\frac{1}{z}\right) dz = 2\pi i \frac{1}{(n+1)!} \qquad (n = 0, 1, 2, \ldots).$$

The final result in part *(a)* thus reduces to

$$\int_C \exp\left(z + \frac{1}{z}\right) dz = 2\pi i \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!}.$$

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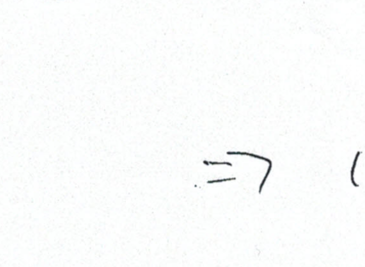
C $\quad I = \int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}_{z=z_k} f(z) \quad (1)$



But at the same time

$I = \int_C f(z)\,dz = -2\pi i \operatorname{Res}_{z=\infty} f(z) \quad (2)$

from sec 71

$\Rightarrow (1) = (2) \Rightarrow \blacksquare$