math 567 HW 08 Solutions

P243

1. (a) From the expansion

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \qquad (|z| < \infty),$$

we see that

$$z\exp\left(\frac{1}{z}\right) = z + 1 + \frac{1}{2!}\cdot\frac{1}{z} + \frac{1}{3!}\cdot\frac{1}{z^2} + \cdots \qquad (0 < |z| < \infty).$$

The principal part of $z\exp\left(\frac{1}{z}\right)$ at the isolated singular point $z = 0$ is, then,

$$\frac{1}{2!}\cdot\frac{1}{z} + \frac{1}{3!}\cdot\frac{1}{z^2} + \cdots;$$

and $z = 0$ is an essential singular point of that function.

(b) The isolated singular point of $\frac{z^2}{1+z}$ is at $z = -1$. Since the principal part at $z = -1$ involves powers of $z + 1$, we begin by observing that

$$z^2 = (z+1)^2 - 2z - 1 = (z+1)^2 - 2(z+1) + 1.$$

This enables us to write

$$\frac{z^2}{1+z} = \frac{(z+1)^2 - 2(z+1) + 1}{z+1} = (z+1) - 2 + \frac{1}{z+1}.$$

Since the principal part is $\frac{1}{z+1}$, the point $z = -1$ is a (simple) pole.

(c) The point $z = 0$ is the isolated singular point of $\frac{\sin z}{z}$, and we can write

$$\frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots \qquad (0 < |z| < \infty).$$

The principal part here is evidently 0, and so $z = 0$ is a removable singular point of the function $\frac{\sin z}{z}$.

2

(*d*) The isolated singular point of $\frac{\cos z}{z}$ is $z=0$. Since

$$\frac{\cos z}{z}=\frac{1}{z}\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\cdots\right)=\frac{1}{z}-\frac{z}{2!}+\frac{z^3}{4!}-\cdots \qquad (0<|z|<\infty),$$

the principal part is $\frac{1}{z}$. This means that $z=0$ is a (simple) pole of $\frac{\cos z}{z}$.

(*e*) Upon writing $\frac{1}{(2-z)^3}=\frac{-1}{(z-2)^3}$, we find that the principal part of $\frac{1}{(2-z)^3}$ at its isolated singular point $z=2$ is simply the function itself. That point is evidently a pole (of order 3).

2. (*a*) The singular point is $z=0$. Since

$$\frac{1-\cosh z}{z^3}=\frac{1}{z^3}\left[1-\left(1+\frac{z^2}{2!}+\frac{z^4}{4!}+\frac{z^6}{6!}+\cdots\right)\right]=-\frac{1}{2!}\cdot\frac{1}{z}-\frac{z}{4!}-\frac{z^3}{6!}-\cdots$$

when $0<|z|<\infty$, we have $m=1$ and $B=-\frac{1}{2!}=-\frac{1}{2}$.

(*b*) Here the singular point is also $z=0$. Since

$$\frac{1-\exp(2z)}{z^4}=\frac{1}{z^4}\left[1-\left(1+\frac{2z}{1!}+\frac{2^2z^2}{2!}+\frac{2^3z^3}{3!}+\frac{2^4z^4}{4!}+\frac{2^5z^5}{5!}+\cdots\right)\right]$$

$$=-\frac{2}{1!}\cdot\frac{1}{z^3}-\frac{2^2}{2!}\cdot\frac{1}{z^2}-\frac{2^3}{3!}\cdot\frac{1}{z}-\frac{2^4}{4!}-\frac{2^5}{5!}z-\cdots$$

when $0<|z|<\infty$, we have $m=3$ and $B=-\frac{2^3}{3!}=-\frac{4}{3}$.

(*c*) The singular point of $\frac{\exp(2z)}{(z-1)^2}$ is $z=1$. The Taylor series

$$\exp(2z)=e^{2(z-1)}e^2=e^2\left[1+\frac{2(z-1)}{1!}+\frac{2^2(z-1)^2}{2!}+\frac{2^3(z-1)^3}{3!}+\cdots\right] \qquad (|z|<\infty)$$

enables us to write the Laurent series

$$\frac{\exp(2z)}{(z-1)^2}=e^2\left[\frac{1}{(z-1)^2}+\frac{2}{1!}\cdot\frac{1}{z-1}+\frac{2^2}{2!}+\frac{2^2}{3!}(z-1)+\cdots\right] \qquad (0<|z-1|<\infty).$$

Thus $m=2$ and $B=e^2\frac{2}{1!}=2e^2$.

3. Since f is analytic at z_0, it has a Taylor series representation

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots \qquad (|z - z_0| < R_0).$$

Let g be defined by means of the equation

$$g(z) = \frac{f(z)}{z - z_0}.$$

(a) Suppose that $f(z_0) \neq 0$. Then

$$g(z) = \frac{1}{z - z_0}\left[f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots\right]$$

$$= \frac{f(z_0)}{z - z_0} + \frac{f'(z_0)}{1!} + \frac{f''(z_0)}{2!}(z - z_0) + \cdots \qquad (0 < |z - z_0| < R_0).$$

This shows that g has a simple pole at z_0, with residue $f(z_0)$.

(b) Suppose, on the other hand, that $f(z_0) = 0$. Then

$$g(z) = \frac{1}{z - z_0}\left[\frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots\right]$$

$$= \frac{f'(z_0)}{1!} + \frac{f''(z_0)}{2!}(z - z_0) + \cdots \qquad (0 < |z - z_0| < R_0).$$

Since the principal part of g at z_0 is just 0, the point $z = 0$ is a removable singular point of g.

5. Write the function

$$f(z) = \frac{8a^3z^2}{(z^2 + a^2)^3} \qquad (a > 0)$$

as

$$f(z) = \frac{\phi(z)}{(z - ai)^3} \quad \text{where} \quad \phi(z) = \frac{8a^3z^2}{(z + ai)^3}.$$

Since the only singularity of $\phi(z)$ is at $z = -ai$, $\phi(z)$ has a Taylor series representation

$$\phi(z) = \phi(ai) + \frac{\phi'(ai)}{1!}(z - ai) + \frac{\phi''(ai)}{2!}(z - ai)^2 + \cdots \qquad (|z - ai| < 2a)$$

about $z = ai$. Thus

$$f(z) = \frac{1}{(z-ai)^3}\left[\phi(ai) + \frac{\phi'(ai)}{1!}(z-ai) + \frac{\phi''(ai)}{2!}(z-ai)^2 + \cdots\right] \quad (0 < |z-ai| < 2a).$$

Now straightforward differentiation reveals that

$$\phi'(z) = \frac{16a^4iz - 8a^3z^2}{(z+ai)^4} \quad \text{and} \quad \phi''(z) = \frac{16a^3(z^2 - 4aiz - a^2)}{(z+ai)^5}.$$

Consequently,

$$\phi(ai) = -a^2i, \quad \phi'(ai) = -\frac{a}{2}, \quad \text{and} \quad \phi''(ai) = -i.$$

This enables us to write

$$f(z) = \frac{1}{(z-ai)^3}\left[-a^2i - \frac{a}{2}(z-ai) - \frac{i}{2}(z-ai)^2 + \cdots\right] \quad (0 < |z-ai| < 2a).$$

The principal part of f at the point $z = ai$ is, then,

$$-\frac{i/2}{z-ai} - \frac{a/2}{(z-ai)^2} - \frac{a^2i}{(z-ai)^3}.$$

p. 248

1. (a) The function $f(z) = \frac{z^2+2}{z-1}$ has an isolated singular point at $z = 1$. Writing $f(z) = \frac{\phi(z)}{z-1}$, where $\phi(z) = z^2 + 2$, and observing that $\phi(z)$ is analytic and nonzero at $z = 1$, we see that $z = 1$ is a pole of order $m = 1$ and that the residue there is $B = \phi(1) = 3$.

(b) If we write

$$f(z) = \left(\frac{z}{2z+1}\right)^3 = \frac{\phi(z)}{\left[z - \left(-\frac{1}{2}\right)\right]^3}, \quad \text{where} \quad \phi(z) = \frac{z^3}{8},$$

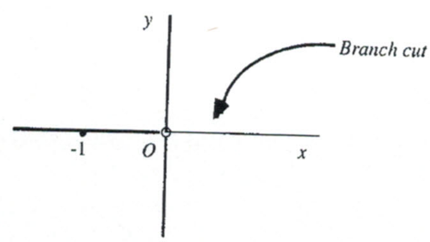
we see that $z = -\frac{1}{2}$ is a singular point of f. Since $\phi(z)$ is analytic and nonzero at that point, f has a pole of order $m = 3$ there. The residue is

$$B = \frac{\phi''(-1/2)}{2!} = -\frac{3}{16}.$$

2. (a) Write the function $f(z) = \dfrac{z^{1/4}}{z+1}$ $(|z| > 0, 0 < \arg z < 2\pi)$ as

$$f(z) = \frac{\phi(z)}{z+1}, \quad \text{where} \quad \phi(z) = z^{1/4} = e^{\frac{1}{4}\log z} \quad (|z| > 0, 0 < \arg z < 2\pi).$$

The function $\phi(z)$ is analytic throughout its domain of definition, indicated in the figure below.



Also,

$$\phi(-1) = (-1)^{1/4} = e^{\frac{1}{4}\log(-1)} = e^{\frac{1}{4}(\ln 1 + i\pi)} = e^{i\pi/4} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1+i}{\sqrt{2}} \neq 0.$$

This shows that the function f has a pole of order $m = 1$ at $z = -1$, the residue there being

$$B = \phi(-1) = \frac{1+i}{\sqrt{2}}.$$

(b) Write the function $f(z) = \dfrac{\operatorname{Log} z}{(z^2+1)^2}$ as

$$f(z) = \frac{\phi(z)}{(z-i)^2} \quad \text{where} \quad \phi(z) = \frac{\operatorname{Log} z}{(z+i)^2}.$$

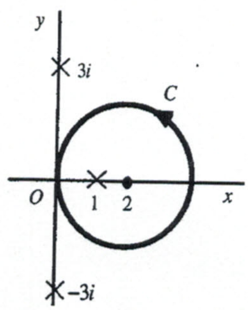
From this, it is clear that $f(z)$ has a pole of order $m = 2$ at $z = i$. Straightforward differentiation then reveals that

$$\operatorname*{Res}_{z=i} \frac{\operatorname{Log} z}{(z^2+1)^2} = \phi'(i) = \frac{\pi + 2i}{8}.$$

3. (a) We wish to evaluate the integral

$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)}\,dz,$$

where C is the circle $|z-2|=2$, taken in the counterclockwise direction. That circle and the singularities $z=1, \pm 3i$ of the integrand are shown in the figure just below.



Observe that the point $z=1$, which is the only singularity inside C, is a simple pole of the integrand and that

$$\operatorname*{Res}_{z=1} \frac{3z^3+2}{(z-1)(z^2+9)} = \left.\frac{3z^3+2}{z^2+9}\right]_{z=1} = \frac{1}{2}.$$

According to the residue theorem, then,

$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)}\,dz = 2\pi i\left(\frac{1}{2}\right) = \pi i.$$

5. Let us evaluate the integral $\int_C \frac{\cosh \pi z\,dz}{z(z^2+1)}$, where C is the positively oriented circle $|z|=2$. All three isolated singularities $z=0, \pm i$ of the integrand are interior to C. The desired residues are

$$\operatorname*{Res}_{z=0} \frac{\cosh \pi z}{z(z^2+1)} = \left.\frac{\cosh \pi z}{z^2+1}\right]_{z=0} = 1,$$

$$\operatorname*{Res}_{z=i} \frac{\cosh \pi z}{z(z^2+1)} = \left.\frac{\cosh \pi z}{z(z+i)}\right]_{z=i} = \frac{1}{2},$$

and

$$\operatorname*{Res}_{z=-i} \frac{\cosh \pi z}{z(z^2+1)} = \left.\frac{\cosh \pi z}{z(z-i)}\right]_{z=-i} = \frac{1}{2}.$$

Consequently,

$$\int_C \frac{\cosh \pi z\,dz}{z(z^2+1)} = 2\pi i\left(1+\frac{1}{2}+\frac{1}{2}\right) = 4\pi i.$$

6. In each part of this problem, C denotes the positively oriented circle $|z| = 3$.

(a) It is straightforward to show that

$$\text{if } f(z) = \frac{(3z+2)^2}{z(z-1)(2z+5)}, \quad \text{then} \quad \frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{(3+2z)^2}{z(1-z)(2+5z)}.$$

This function $\frac{1}{z^2} f\left(\frac{1}{z}\right)$ has a simple pole at $z = 0$, and

$$\int_C \frac{(3z+2)^2}{z(z-1)(2z+5)} dz = 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] = 2\pi i \left(\frac{9}{2}\right) = 9\pi i.$$

p 255

1 a we use that

$$\csc z = \frac{1}{\sin z} = \frac{p(z)}{q(z)}, \text{ where } p(z) = 1, \quad q(z) = \sin z$$

~~Because~~ Since $p(0) = 1 \neq 0$ while $q(0) = \sin 0 = 0$ but

$$q'(0) = \cos 0 = 1 \neq 0$$

$\Rightarrow$ $z = 0$ is the simple pole of $\frac{1}{\sin z}$

$$\Rightarrow \operatorname*{Res}_{z=0} \frac{1}{\sin z} = \frac{p(0)}{q'(0)} = \frac{1}{1} = 1$$

b From Ex. ?, Sec 67 we use that

$$\csc z = \frac{1}{z} + \frac{1}{3!} z + \left[\frac{1}{(3!)^2} - \frac{1}{5!} \right] z^3 + \cdots, \quad 0 < |z| < \pi$$

[to show that: $\frac{1}{\sin z} = \frac{1}{z - \frac{z^3}{3!} + \cdots} = \frac{1}{z\left(1 - \frac{z^2}{3!} + \cdots\right)} = \frac{1}{z}\left(1 + \frac{z^2}{3!} + \cdots\right)$]

Since the coefficient of $\frac{1}{z}$ here is 1

$\Rightarrow$ $z = 0$ is a simple pole of $\csc z$

with $\operatorname*{Res}_{z=0} \csc z = 1$

2. (a) Write

$$\frac{z-\sinh z}{z^2\sinh z}=\frac{p(z)}{q(z)}, \quad \text{where} \quad p(z)=z-\sinh z \text{ and } q(z)=z^2\sinh z.$$

Since

$$p(\pi i)=\pi i\neq 0, \quad q(\pi i)=0, \quad \text{and} \quad q'(\pi i)=\pi^2\neq 0,$$

it follows that

$$\operatorname*{Res}_{z=\pi i}\frac{z-\sinh z}{z^2\sinh z}=\frac{p(\pi i)}{q'(\pi i)}=\frac{\pi i}{\pi^2}=\frac{i}{\pi}.$$

3. (a) Write

$$f(z)=\frac{p(z)}{q(z)}, \quad \text{where} \quad p(z)=z \text{ and } q(z)=\cos z.$$

Observe that

$$q\left(\frac{\pi}{2}+n\pi\right)=0 \qquad (n=0,\pm1,\pm2,\ldots).$$

Also, for the stated values of n,

$$p\left(\frac{\pi}{2}+n\pi\right)=\frac{\pi}{2}+n\pi\neq 0 \quad \text{and} \quad q'\left(\frac{\pi}{2}+n\pi\right)=-\sin\left(\frac{\pi}{2}+n\pi\right)=(-1)^{n+1}\neq 0.$$

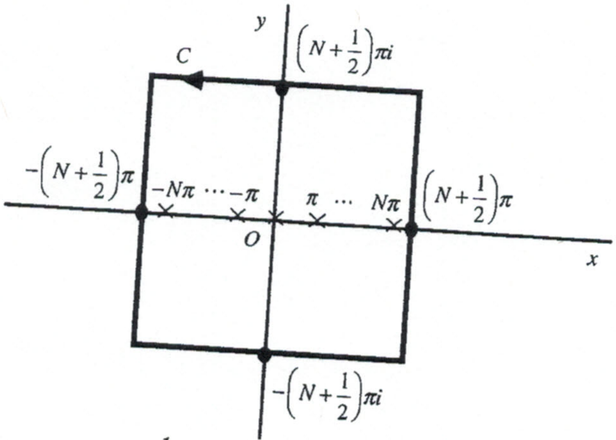
So the function $f(z)=\dfrac{z}{\cos z}$ has poles of order $m=1$ at each of the points

$$z_n=\frac{\pi}{2}+n\pi \qquad (n=0,\pm1,\pm2,\ldots).$$

The corresponding residues are

$$B=\frac{p(z_n)}{q'(z_n)}=(-1)^{n+1}z_n.$$

5. The simple closed contour C_N is as shown in the figure below.



Within C_N, the function $\dfrac{1}{z^2 \sin z}$ has isolated singularities at

$$z=0 \quad \text{and} \quad z=\pm n\pi \ (n=1,2,\ldots,N).$$

To find the residue at $z=0$, we recall the Laurent series for $\csc z$ that was found in Exercise 2, Sec. 67, and write

$$\frac{1}{z^2 \sin z}=\frac{1}{z^2}\csc z=\frac{1}{z^2}\left\{\frac{1}{z}+\frac{1}{3!}z+\left[\frac{1}{(3!)^2}-\frac{1}{5!}\right]z^3+\cdots\right\}$$

$$=\frac{1}{z^3}+\frac{1}{6}\cdot\frac{1}{z}+\left[\frac{1}{(3!)^2}-\frac{1}{5!}\right]z+\cdots \qquad (0<|z|<\pi).$$

This tells us that $\dfrac{1}{z^2 \sin z}$ has a pole of order 3 at $z=0$ and that

$$\operatorname*{Res}_{z=0}\frac{1}{z^2 \sin z}=\frac{1}{6}.$$

As for the points $z=\pm n\pi$ $(n=1,2,\ldots,N)$, write

$$\frac{1}{z^2 \sin z}=\frac{p(z)}{q(z)}, \quad \text{where} \quad p(z)=1 \text{ and } q(z)=z^2 \sin z.$$

Since

$$p(\pm n\pi) = 1 \neq 0, \quad q(\pm n\pi) = 0, \quad \text{and} \quad q'(\pm n\pi) = n^2\pi^2 \cos n\pi = (-1)^n n^2\pi^2 \neq 0,$$

it follows that

$$\operatorname*{Res}_{z=\pm n\pi} \frac{1}{z^2 \sin z} = \frac{1}{(-1)^n n^2\pi^2} \cdot \frac{(-1)^n}{(-1)^n} = \frac{(-1)^n}{n^2\pi^2}.$$

So, by the residue theorem,

$$\int_{C_N} \frac{dz}{z^2 \sin z} dz = 2\pi i \left[\frac{1}{6} + 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2} \right].$$

Rewriting this equation in the form

$$\sum_{n=1}^{N} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12} - \frac{\pi}{4i} \int_{C_N} \frac{dz}{z^2 \sin z}$$

and recalling from Exercise 8, Sec. 43, that the value of the integral here tends to zero as N tends to infinity, we arrive at the desired summation formula:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

6. The path C here is the positively oriented boundary of the rectangle with vertices at the points ± 2 and $\pm 2 + i$. The problem is to evaluate the integral

$$\int_C \frac{dz}{(z^2-1)^2+3}.$$

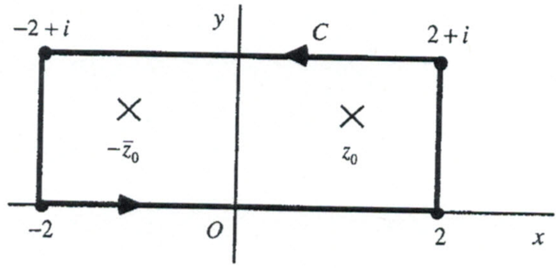
The isolated singularities of the integrand are the zeros of the polynomial

$$q(z) = (z^2-1)^2 + 3.$$

Setting this polynomial equal to zero and solving for z^2, we find that any zero z of $q(z)$ has the property $z^2 = 1 \pm \sqrt{3}i$. It is straightforward to find the two square roots of $1+\sqrt{3}i$ and also the two square roots of $1-\sqrt{3}i$. These are the four zeros of $q(z)$. Only two of those zeros,

$$z_0 = \sqrt{2}e^{i\pi/6} = \frac{\sqrt{3}+i}{\sqrt{2}} \quad \text{and} \quad -\overline{z}_0 = -\sqrt{2}e^{-i\pi/6} = \frac{-\sqrt{3}+i}{\sqrt{2}},$$

lie inside C. They are shown in the figure below.



To find the residues at z_0 and $-\bar{z}_0$, we write the integrand of the integral to be evaluated as

$$\frac{1}{(z^2-1)^2+3} = \frac{p(z)}{q(z)}, \quad \text{where} \quad p(z)=1 \text{ and } q(z)=(z^2-1)^2+3.$$

This polynomial $q(z)$ is, of course, the same $q(z)$ as above; hence $q(z_0)=0$. Note, too, that p and q are analytic at z_0 and that $p(z_0)\neq 0$. Finally, it is straightforward to show that $q'(z)=4z\left(z^2-1\right)$ and hence that

$$q'(z_0)=4z_0\left(z_0^2-1\right)=-2\sqrt{6}+6\sqrt{2}i\neq 0.$$

We may conclude, then, that z_0 is a simple pole of the integrand, with residue

$$\frac{p(z_0)}{q'(z_0)}=\frac{1}{-2\sqrt{6}+6\sqrt{2}i}.$$

Similar results are to be found at the singular point $-\bar{z}_0$. To be specific, it is easy to see that

$$q'(-\bar{z}_0)=-q'(\bar{z}_0)=-\overline{q'(z_0)}=2\sqrt{6}+6\sqrt{2}i\neq 0,$$

the residue of the integrand at $-\bar{z}_0$ being

$$\frac{p(-\bar{z}_0)}{q'(-\bar{z}_0)}=\frac{1}{2\sqrt{6}+6\sqrt{2}i}.$$

Finally, by the residue theorem,

$$\int_C \frac{dz}{(z^2-1)^2+3}=2\pi i\left(\frac{1}{-2\sqrt{6}+6\sqrt{2}i}+\frac{1}{2\sqrt{6}+6\sqrt{2}i}\right)=\frac{\pi}{2\sqrt{2}}.$$

7. We are given that $f(z)=1/[q(z)]^2$, where q is analytic at z_0, $q(z_0)=0$, and $q'(z_0)\neq 0$. These conditions on q tell us that q has a zero of order $m=1$ at z_0. Hence $q(z)=(z-z_0)g(z)$, where g is a function that is analytic and nonzero at z_0; and this enables us to write

$$f(z)=\frac{\phi(z)}{(z-z_0)^2}, \quad \text{where} \quad \phi(z)=\frac{1}{[g(z)]^2}.$$

So f has a pole of order 2 at z_0, and

$$\operatorname*{Res}_{z=z_0} f(z)=\phi'(z_0)=-\frac{2g'(z_0)}{[g(z_0)]^3}.$$

But, since $q(z)=(z-z_0)g(z)$, we know that

$$q'(z)=(z-z_0)g'(z)+g(z) \quad \text{and} \quad q''(z)=(z-z_0)g''(z)+2g'(z).$$

Then, by setting $z=z_0$ in these last two equations, we find that

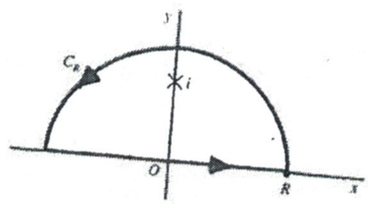
$$q'(z_0)=g(z_0) \quad \text{and} \quad q''(z_0)=2g'(z_0).$$

Consequently, our expression for the residue of f at z_0 can be put in the desired form:

$$\operatorname*{Res}_{z=0} f(z)=-\frac{q''(z_0)}{[q'(z_0)]^3}.$$

p. 267

1. To evaluate the integral $\int_0^\infty \frac{dx}{x^2+1}$, we integrate the function $f(z)=\frac{1}{z^2+1}$ around the simple closed contour shown below, where $R>1$.



We see that

$$\int_{-R}^{R}\frac{dx}{x^2+1}+\int_{C_R}\frac{dz}{z^2+1}=2\pi i B,$$

where

$$B=\operatorname*{Res}_{z=i}\frac{1}{z^2+1}=\operatorname*{Res}_{z=i}\frac{1}{(z-i)(z+i)}=\left.\frac{1}{z+i}\right]_{z=i}=\frac{1}{2i}.$$

Thus

$$\int_{-R}^{R}\frac{dx}{x^2+1}=\pi-\int_{C_R}\frac{dz}{z^2+1}.$$

Now if z is a point on C_R,

$$|z^2+1|\ge||z|^2-1|=R^2-1;$$

and so

$$\left|\int_{C_R}\frac{dz}{z^2+1}\right|\le\frac{\pi R}{R^2-1}=\frac{\frac{\pi}{R}}{1-\frac{1}{R^2}}\to 0 \quad\text{as}\quad R\to\infty.$$

Finally, then

$$\int_{-\infty}^{\infty}\frac{dx}{x^2+1}=\pi,\quad\text{or}\quad\int_0^\infty\frac{dx}{x^2+1}=\frac{\pi}{2}.$$

2. The integral $\int_0^\infty \frac{dx}{(x^2+1)^2}$ can be evaluated using the function $f(z)=\frac{1}{(z^2+1)^2}$ and the same simple closed contour as in Exercise 1. Here

$$\int_{-R}^{R} \frac{dx}{(x^2+1)^2} + \int_{C_R} \frac{dz}{(z^2+1)^2} = 2\pi i B,$$

where $B = \operatorname*{Res}_{z=i} \frac{1}{(z^2+1)^2}$. Since

$$\frac{1}{(z^2+1)^2} = \frac{\phi(z)}{(z-i)^2}, \quad \text{where} \quad \phi(z) = \frac{1}{(z+i)^2},$$

we readily find that $B = \phi'(i) = \frac{1}{4i}$, and so

$$\int_{-R}^{R} \frac{dx}{(x^2+1)^2} = \frac{\pi}{2} - \int_{C_R} \frac{dz}{(z^2+1)^2}.$$

If z is a point on C_R, we know from Exercise 1 that

$$|z^2+1| \ge R^2 - 1;$$

thus

$$\left| \int_{C_R} \frac{dz}{(z^2+1)^2} \right| \le \frac{\pi R}{(R^2-1)^2} = \frac{\frac{\pi}{R^3}}{\left(1-\frac{1}{R^2}\right)^2} \to 0 \quad \text{as} \quad R \to \infty.$$

The desired result is, then,

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{2}, \quad \text{or} \quad \int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}.$$

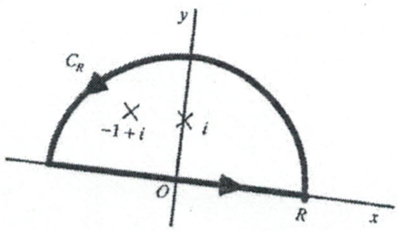
7. In order to show that

$$\text{P.V.}\int_{-\infty}^{\infty}\frac{x\,dx}{(x^2+1)(x^2+2x+2)}=-\frac{\pi}{5},$$

we introduce the function

$$f(z)=\frac{z}{(z^2+1)(z^2+2z+2)}$$

and the simple closed contour shown below.



Observe that the singularities of $f(z)$ are at i, $z_0=-1+i$ and their conjugates $-i$, $\bar{z}_0=-1-i$ in the lower half plane. Also, if $R>\sqrt{2}$, we see that

$$\int_{-R}^{R} f(x)\,dx+\int_{C_R} f(z)\,dz=2\pi i(B_0+B_1),$$

where

$$B_0=\operatorname*{Res}_{z=z_0} f(z)=\left[\frac{z}{(z^2+1)(z-\bar{z}_0)}\right]_{z=z_0}=-\frac{1}{10}+\frac{3}{10}i$$

and

$$B_1=\operatorname*{Res}_{z=i} f(z)=\left[\frac{z}{(z+i)(z^2+2z+2)}\right]_{z=i}=\frac{1}{10}-\frac{1}{5}i.$$

Evidently, then,

$$\int_{-R}^{R}\frac{x\,dx}{(x^2+1)(x^2+2x+2)}=-\frac{\pi}{5}-\int_{C_R}\frac{z\,dz}{(z^2+1)(z^2+2z+2)}.$$

Since

$$\left|\int_{C_R}\frac{z\,dz}{(z^2+1)(z^2+2z+2)}\right|=\left|\int_{C_R}\frac{z\,dz}{(z^2+1)(z-z_0)(z-\bar{z}_0)}\right|\le\frac{\pi R^2}{(R^2-1)(R-\sqrt{2})^2}\to 0$$

as $R\to\infty$, this means that

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{x\,dx}{(x^2+1)(x^2+2x+2)}=-\frac{\pi}{5}.$$

This is the desired result.

17

9. Let m and n be integers, where $0 \le m < n$. The problem here is to derive the integration formula

$$\int_0^\infty \frac{x^{2m}}{x^{2n}+1}\,dx = \frac{\pi}{2n}\csc\left(\frac{2m+1}{2n}\pi\right).$$

(a) The zeros of the polynomial $z^{2n}+1$ occur when $z^{2n}=-1$. Since

$$(-1)^{1/(2n)} = \exp\left[i\frac{(2k+1)\pi}{2n}\right] \qquad (k=0,1,2,\ldots,2n-1),$$

it is clear that the zeros of $z^{2n}+1$ in the upper half plane are

$$c_k = \exp\left[i\frac{(2k+1)\pi}{2n}\right] \qquad (k=0,1,2,\ldots,n-1)$$

and that there are none on the real axis.

(b) With the aid of Theorem 2 in Sec. 76, we find that

$$\operatorname*{Res}_{z=c_k}\frac{z^{2m}}{z^{2n}+1} = \frac{c_k^{2m}}{2n c_k^{2n-1}} = \frac{1}{2n}c_k^{2(m-n)+1} \qquad (k=0,1,2,\ldots,n-1).$$

Putting $\alpha = \dfrac{2m+1}{2n}\pi$, we can write

$$c_k^{2(m-n)+1} = \exp\left[i\frac{(2k+1)\pi(2m-2n+1)}{2n}\right]$$

$$= \exp\left[i\frac{(2k+1)(2m+1)\pi}{2n}\right]\exp[-i(2k+1)\pi] = -e^{i(2k+1)\alpha}.$$

Thus

$$\operatorname*{Res}_{z=c_k}\frac{z^{2m}}{z^{2n}+1} = -\frac{1}{2n}e^{i(2k+1)\alpha} \qquad (k=0,1,2,\ldots,n-1).$$

In view of the identity (see Exercise 9, Sec. 8)

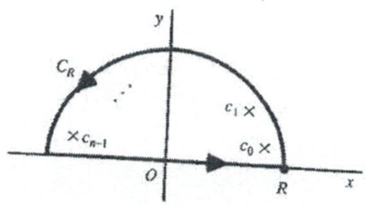
$$\sum_{k=0}^{n-1} z^k = \frac{1-z^n}{1-z} \qquad (z \ne 1),$$

then,

$$2\pi i\sum_{k=0}^{n-1}\operatorname{Res}_{z=c_k}\frac{z^{2m}}{z^{2n}+1} = -\frac{\pi i}{n}e^{i\alpha}\sum_{k=0}^{n-1}(e^{i2\alpha})^k = -\frac{\pi i}{n}e^{i\alpha}\frac{1-e^{i2\alpha n}}{1-e^{i2\alpha}}\cdot\frac{e^{-i\alpha}}{e^{-i\alpha}} = -\frac{\pi i}{n}\cdot\frac{e^{i2\alpha n}-1}{e^{i\alpha}-e^{-i\alpha}}$$

$$= -\frac{\pi i}{n}\cdot\frac{e^{i(2m+1)\pi}-1}{e^{i\alpha}-e^{-i\alpha}} = \frac{\pi}{n}\cdot\frac{2i}{e^{i\alpha}-e^{-i\alpha}} = \frac{\pi}{n\sin\alpha}.$$

(c) Consider the path shown below, where $R>1$.



The residue theorem tells us that

$$\int_{-R}^{R}\frac{x^{2m}}{x^{2n}+1}dx + \int_{C_R}\frac{z^{2m}}{z^{2n}+1}dz = 2\pi i\sum_{k=0}^{n-1}\operatorname{Res}_{z=c_k}\frac{z^{2m}}{z^{2n}+1},$$

or

$$\int_{-R}^{R}\frac{x^{2m}}{x^{2n}+1}dx = \frac{\pi}{n\sin\alpha} - \int_{C_R}\frac{z^{2m}}{z^{2n}+1}dz.$$

Observe that if z is a point on C_R, then

$$|z^{2m}| = R^{2m} \quad\text{and}\quad |z^{2n}+1| \ge R^{2n}-1.$$

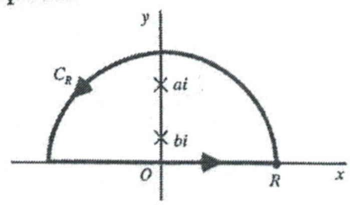
Consequently,

$$\left|\int_{C_R}\frac{z^{2m}}{z^{2n}+1}dz\right| \le \frac{R^{2m}}{R^{2n}-1}\pi R\cdot\frac{R^{-2n}}{R^{-2n}} = \pi\,\frac{\dfrac{1}{R^{2(n-m)-1}}}{1-\dfrac{1}{R^{2n}}} \to 0;$$

and the desired integration formula follows.

p. 275

1. The problem here is to evaluate the integral $\int_{-\infty}^{\infty}\frac{\cos x\,dx}{(x^2+a^2)(x^2+b^2)}$, where $a>b>0$. To do this, we introduce the function $f(z)=\frac{1}{(z^2+a^2)(z^2+b^2)}$, whose singularities ai and bi lie inside the simple closed contour shown below, where $R>a$. The other singularities are, of course, in the lower half plane.



According to the residue theorem,

$$\int_{-R}^{R}\frac{e^{ix}\,dx}{(x^2+a^2)(x^2+b^2)}+\int_{C_R}f(z)e^{iz}dz=2\pi i(B_1+B_2),$$

where

$$B_1=\operatorname*{Res}_{z=ai}[f(z)e^{iz}]=\left[\frac{e^{iz}}{(z+ai)(z^2+b^2)}\right]_{z=ai}=\frac{e^{-a}}{2a(b^2-a^2)i}$$

and

$$B_2=\operatorname*{Res}_{z=bi}[f(z)e^{iz}]=\left[\frac{e^{iz}}{(z^2+a^2)(z+bi)}\right]_{z=bi}=\frac{e^{-b}}{2b(a^2-b^2)i}.$$

That is,

$$\int_{-R}^{R}\frac{e^{ix}\,dx}{(x^2+a^2)(x^2+b^2)}=\frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b}-\frac{e^{-a}}{a}\right)-\int_{C_R}f(z)e^{iz}dz,$$

or

$$\int_{-R}^{R}\frac{\cos x\,dx}{(x^2+a^2)(x^2+b^2)}=\frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b}-\frac{e^{-a}}{a}\right)-\operatorname{Re}\int_{C_R}f(z)e^{iz}dz.$$

Now, if z is a point on C_R,

$$|f(z)|\le M_R \quad\text{where}\quad M_R=\frac{1}{(R^2-a^2)(R^2-b^2)}$$

and $|e^{iz}|=e^{-y}\le 1$. Hence

$$\left|\operatorname{Re}\int_{C_R}f(z)e^{iz}dz\right|\le\left|\int_{C_R}f(z)e^{iz}dz\right|\le M_R\pi R=\frac{\pi R}{(R^2-a^2)(R^2-b^2)}\to 0 \text{ as } R\to\infty.$$

So it follows that

$$\int_{-\infty}^{\infty}\frac{\cos x\,dx}{(x^2+a^2)(x^2+b^2)}=\frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b}-\frac{e^{-a}}{a}\right)\qquad(a>b>0).$$

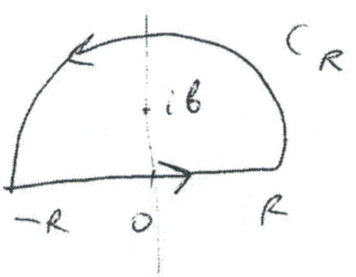
3

$$I = \int_0^\infty \frac{\cos ax}{(x^2+b^2)^2}\,dx \qquad (a>0,\ b>0)$$

$$I_0 = \int_{-\infty}^{\infty} \frac{e^{iax}}{(x^2+b^2)^2}\,dx = 2I$$

$$f(z) = \frac{e^{iaz}}{(z^2+b^2)^2}$$

pole of second order



$$\left(\int_{-R}^{R} + \int_{C_R}\right) f(z)\,dz = 2\pi i \operatorname*{Res}_{z=ib} f(z)$$

$$= 2\pi i \operatorname*{Res}_{z=ib} \frac{e^{iaz}}{(z+ib)^2(z-ib)^2} = 2\pi i \left(\frac{e^{iaz}}{(z+ib)^2}\right)'\Bigg|_{z=ib}$$

$$= 2\pi i\, \frac{ia e^{iaz}(z+ib)^2 - e^{iaz}\,2(z+ib)}{(z+ib)^4}\Bigg|_{z=ib}$$

$$= 2\pi i\, \frac{e^{-ab}\left[ia(2ib)^2 - 2\cdot 2ib\right]}{(2ib)^4} = -\frac{2\pi i\cdot 2}{(2ib)^3}[1+ab]e^{-ab}$$

$$= \frac{\pi}{2} e^{-ab}\, \frac{1+ab}{b^3}$$

$$\lim_{R\to\infty} \int_{C_R} F(z)\,dz = 0 \text{ by Jordan's Lemma.}$$

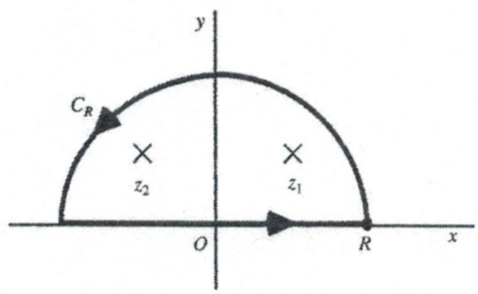
$$\Rightarrow I = \int_0^\infty \frac{\cos ax}{(x^2+b^2)^2}\,dx = \frac{I_0}{2}$$

$$= \frac{1}{2} \cdot \frac{\pi}{2} e^{-ab} \frac{1+ab}{b^3} = \frac{\pi}{4b^3} e^{-ab}(1+ab)$$

6. The integral to be evaluated is $\int_{-\infty}^{\infty} \frac{x^3 \sin ax}{x^4+4}\,dx$, where $a > 0$. We define the function $f(z) = \frac{z^3}{z^4+4}$; and, by computing the fourth roots of -4, we find that the singularities

$$z_1 = \sqrt{2}e^{i\pi/4} = 1+i \quad \text{and} \quad z_2 = \sqrt{2}e^{i3\pi/4} = \sqrt{2}e^{i\pi/4}e^{i\pi/2} = (1+i)i = -1+i$$

both lie inside the simple closed contour shown below, where $R > \sqrt{2}$. The other two singularities lie below the real axis.



The residue theorem and the method of Theorem 2 in Sec. 76 for finding residues at simple poles tell us that

$$\int_{-R}^{R} \frac{x^3 e^{iax}}{x^4+4}\,dx + \int_{C_R} f(z)e^{iaz}\,dz = 2\pi i(B_1 + B_2),$$

where

$$B_1 = \operatorname*{Res}_{z=z_1} \frac{z^3 e^{iaz}}{z^4+4} = \frac{z_1^3 e^{iaz_1}}{4z_1^3} = \frac{e^{iaz_1}}{4} = \frac{e^{ia(1+i)}}{4} = \frac{e^{-a}e^{ia}}{4}$$

and

$$B_2 = \operatorname*{Res}_{z=z_2} \frac{z^3 e^{iaz}}{z^4+4} = \frac{z_2^3 e^{iaz_2}}{4z_2^3} = \frac{e^{iaz_2}}{4} = \frac{e^{ia(-1+i)}}{4} = \frac{e^{-a}e^{-ia}}{4}.$$

Since

$$2\pi i(B_1 + B_2) = \pi i e^{-a}\left(\frac{e^{ia}+e^{-ia}}{2}\right) = i\pi e^{-a}\cos a,$$

we are now able to write

$$\int_{-R}^{R} \frac{x^3 \sin ax}{x^4+4}\,dx = \pi e^{-a}\cos a - \operatorname{Im}\int_{C_R} f(z)e^{iaz}\,dz.$$

Furthermore, if z is a point on C_R, then

$$|f(z)| \le M_R \quad \text{where} \quad M_R = \frac{R^3}{R^4-4} \to 0 \text{ as } R \to \infty;$$

and this means that

$$\left|\operatorname{Im}\int_{C_R} f(z)e^{iaz}\,dz\right| \le \left|\int_{C_R} f(z)e^{iaz}\,dz\right| \to 0 \text{ as } R \to \infty,$$

according to limit (1), Sec. 74. Finally, then,

$$\int_{-\infty}^{\infty} \frac{x^3 \sin ax}{x^4+4}\,dx = \pi e^{-a}\cos a \qquad (a > 0).$$