math 561 HW 09 Solutions

p. 275 ①

(10) To find the Cauchy principal value of the improper integral $\int_{-\infty}^{\infty}\frac{(x+1)\cos x}{x^2+4x+5}dx$, we shall use the function $f(z)=\frac{z+1}{z^2+4z+5}=\frac{z+1}{(z-z_1)(z-\bar{z}_1)}$, where $z_1=-2+i$, and $\bar{z}_1=-2-1$, and the same simple closed contour as in Exercise 9. In this case,

$$\int_{-R}^{R}\frac{(x+1)e^{ix}\,dx}{x^2+4x+5}+\int_{C_R}f(z)e^{iz}dz=2\pi iB,$$

where

$$B=\operatorname*{Res}_{z=z_1}\left[\frac{(z+1)e^{iz}}{(z-z_1)(z-\bar{z}_1)}\right]=\frac{(z_1+1)e^{iz_1}}{(z-\bar{z}_1)}=\frac{(-1+i)e^{-2i}}{2ei}.$$

Thus

$$\int_{-R}^{R}\frac{(x+1)\cos x}{x^2+4x+5}dx=\operatorname{Re}(2\pi iB)-\int_{C_R}f(z)e^{iz},$$

or

$$\int_{-R}^{R}\frac{(x+1)\cos x}{x^2+4x+5}dx=\frac{\pi}{e}(\sin 2-\cos 2)-\int_{C_R}f(z)e^{iz}dz.$$

Finally, we observe that if z is a point on C_R, then

$$|f(z)|\le M_R \quad \text{where} \quad M_R=\frac{R+1}{(R-|z_1|)(R-|\bar{z}_1|)}=\frac{R+1}{(R-\sqrt{5})^2}\to 0 \text{ as } R\to\infty.$$

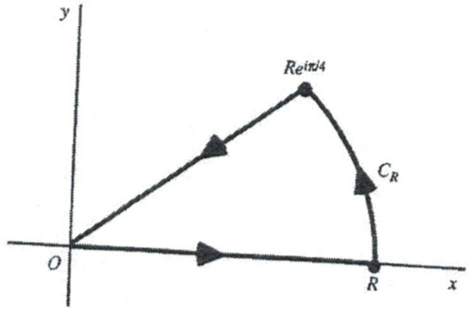
The theorem in Sec. 81 then tells us that

$$\left|\operatorname{Re}\int_{C_R}f(z)e^{iz}dz\right|\le\left|\int_{C_R}f(z)e^{iz}dz\right|\to 0 \text{ as } R\to\infty,$$

and so

$$\text{P.V.}\int_{-\infty}^{\infty}\frac{(x+1)\cos x}{x^2+4x+5}dx=\frac{\pi}{e}(\sin 2-\cos 2).$$

12. (a) Since the function $f(z) = \exp(iz^2)$ is entire, the Cauchy-Goursat theorem tells us that its integral around the positively oriented boundary of the sector $0 \le r \le R,\ 0 \le \theta \le \pi/4$ has value zero. The closed path is shown below.



A parametric representation of the horizontal line segment from the origin to the point R is $z = x\ (0 \le x \le R)$, and a representation for the segment from the origin to the point $Re^{i\pi/4}$ is $z = re^{i\pi/4}\ (0 \le r \le R)$. Thus

$$\int_0^R e^{ix^2}\,dx + \int_{C_R} e^{iz^2}\,dz - e^{i\pi/4}\int_0^R e^{-r^2}\,dr = 0,$$

or

$$\int_0^R e^{ix^2}\,dx = e^{i\pi/4}\int_0^R e^{-r^2}\,dr - \int_{C_R} e^{iz^2}\,dz.$$

By equating real parts and then imaginary parts on each side of this last equation, we see that

$$\int_0^R \cos(x^2)\,dx = \frac{1}{\sqrt{2}}\int_0^R e^{-r^2}\,dr - \operatorname{Re}\int_{C_R} e^{iz^2}\,dz$$

and

$$\int_0^R \sin(x^2)\,dx = \frac{1}{\sqrt{2}}\int_0^R e^{-r^2}\,dr - \operatorname{Im}\int_{C_R} e^{iz^2}\,dz.$$

(b) A parametric representation for the arc C_R is $z = Re^{i\theta}\ (0 \le \theta \le \pi/4)$. Hence

$$\int_{C_R} e^{iz^2}\,dz = \int_0^{\pi/4} e^{iR^2e^{i2\theta}}\,Rie^{i\theta}\,d\theta = iR\int_0^{\pi/4} e^{-R^2\sin 2\theta}e^{iR^2\cos 2\theta}e^{i\theta}\,d\theta.$$

Since $\left|e^{iR^2\cos 2\theta}\right| = 1$ and $\left|e^{i\theta}\right| = 1$, it follows that

$$\left|\int_{C_R} e^{iz^2}\,dz\right| \le R\int_0^{\pi/4} e^{-R^2\sin 2\theta}\,d\theta.$$

Then, by making the substitution $\phi = 2\theta$ in this last integral and referring to the form (2), Sec. 81, of Jordan's inequality, we find that

$$\left|\int_{C_R} e^{iz^2}\,dz\right| \le \frac{R}{2}\int_0^{\pi/2} e^{-R^2\sin\phi}\,d\phi \le \frac{R}{2}\cdot\frac{\pi}{2R^2} = \frac{\pi}{4R} \to 0 \text{ as } R \to \infty.$$

3

(*c*) In view of the result in part (*b*) and the integration formula

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2},$$

it follows from the last two equations in part (*a*) that

$$\int_0^\infty \cos(x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}} \quad \text{and} \quad \int_0^\infty \sin(x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

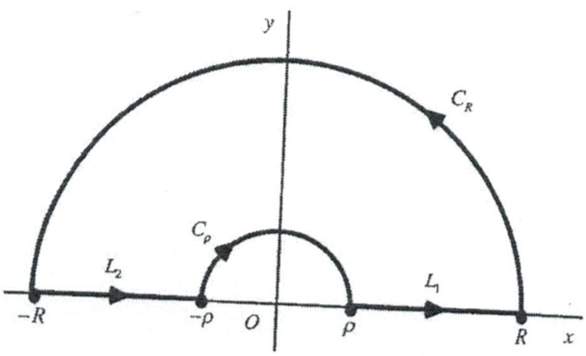
4

p. 286

1. The main problem here is to derive the integration formula

$$\int_0^\infty \frac{\cos(ax)-\cos(bx)}{x^2}\,dx = \frac{\pi}{2}(b-a) \qquad (a \ge 0, b \ge 0),$$

using the indented contour shown below.



Applying the Cauchy-Goursat theorem to the function

$$f(z) = \frac{e^{iaz}-e^{ibz}}{z^2},$$

we have

$$\int_{L_1} f(z)\,dz + \int_{C_R} f(z)\,dz + \int_{L_2} f(z)\,dz + \int_{C_\rho} f(z)\,dz = 0,$$

or

$$\int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz = -\int_{C_\rho} f(z)\,dz - \int_{C_R} f(z)\,dz.$$

Since L_1 and $-L_2$ have parametric representations

$$L_1 : z = re^{i0} = r \; (\rho \le r \le R) \quad \text{and} \quad -L_2 : z = re^{i\pi} = -r \; (\rho \le r \le R),$$

we can see that

$$\int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz = \int_{L_1} f(z)\,dz - \int_{-L_2} f(z)\,dz = \int_\rho^R \frac{e^{iar}-e^{ibr}}{r^2}\,dr + \int_\rho^R \frac{e^{-iar}-e^{-ibr}}{r^2}\,dr$$

$$= \int_\rho^R \frac{(e^{iar}+e^{-iar})-(e^{ibr}+e^{-ibr})}{r^2}\,dr = 2\int_\rho^R \frac{\cos(ar)-\cos(br)}{r^2}\,dr.$$

Thus

$$2\int_{\rho}^{R} \frac{\cos(ar)-\cos(br)}{r^2}\,dr = -\int_{C_\rho} f(z)\,dz - \int_{C_R} f(z)\,dz.$$

In order to find the limit of the first integral on the right here as $\rho \to 0$, we write

$$f(z) = \frac{1}{z^2}\left[\left(1+\frac{iaz}{1!}+\frac{(iaz)^2}{2!}+\frac{(iaz)^3}{3!}+\cdots\right)-\left(1+\frac{ibz}{1!}+\frac{(ibz)^2}{2!}+\frac{(ibz)^3}{3!}+\cdots\right)\right]$$

$$= \frac{i(a-b)}{z}+\cdots \quad (0<|z|<\infty).$$

From this we see that $z=0$ is a simple pole of $f(z)$, with residue $B_0 = i(a-b)$. Thus

$$\lim_{\rho\to 0}\int_{C_\rho} f(z)\,dz = -B_0\pi i = -i(a-b)\pi i = \pi(a-b).$$

As for the limit of the value of the second integral as $R \to \infty$, we note that if z is a point on C_R, then

$$f(z) \le \frac{|e^{iaz}|+|e^{ibz}|}{|z|^2} = \frac{e^{-ay}+e^{-by}}{R^2} \le \frac{1+1}{R^2} = \frac{2}{R^2}.$$

Consequently,

$$\left|\int_{C_R} f(z)\,dz\right| \le \frac{2}{R^2}\pi R = \frac{2\pi}{R} \to 0 \text{ as } R \to \infty.$$

It is now clear that letting $\rho \to 0$ and $R \to \infty$ yields

$$2\int_{0}^{\infty} \frac{\cos(ar)-\cos(br)}{r^2}\,dr = \pi(b-a).$$

This is the desired integration formula, with the variable of integration r instead of x. Observe that when $a=0$ and $b=2$, that result becomes

$$\int_{0}^{\infty} \frac{1-\cos(2x)}{x^2}\,dx = \pi.$$

But $\cos(2x) = 1-2\sin^2 x$, and we arrive at

$$\int_{0}^{\infty} \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}.$$

page 226

6

4. Let us use the function

$$f(z) = \frac{(\log z)^2}{z^2+1} \qquad \left(|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right)$$

and the contour in Exercise 2 to show that

$$\int_0^\infty \frac{(\ln x)^2}{x^2+1}\,dx = \frac{\pi^3}{8} \quad \text{and} \quad \int_0^\infty \frac{\ln x}{x^2+1}\,dx = 0.$$

Integrating $f(z)$ around the closed path shown in Exercise 2, we have

$$\int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=i} f(z) - \int_{C_\rho} f(z)\,dz - \int_{C_R} f(z)\,dz.$$

Since

$$f(z) = \frac{\phi(z)}{z-i} \quad \text{where} \quad \phi(z) = \frac{(\log z)^2}{z+i},$$

the point $z = i$ is a simple pole of $f(z)$ and the residue is

$$\operatorname*{Res}_{z=i} f(z) = \phi(i) = \frac{(\log i)^2}{2i} = \frac{(\ln 1 + i\pi/2)^2}{2i} = -\frac{\pi^2}{8i}.$$

Also, the parametric representations

$$L_1 : z = re^{i0} = r \ (\rho \le r \le R) \quad \text{and} \quad -L_2 : z = re^{i\pi} = -r \ (\rho \le r \le R)$$

enable us to write

$$\int_{L_1} f(z)dz = \int_\rho^R \frac{(\ln r)^2}{r^2+1}\,dr \quad \text{and} \quad \int_{L_2} f(z)dz = \int_\rho^R \frac{(\ln r + i\pi)^2}{r^2+1}\,dr.$$

Since

$$\int_{L_1} f(z)dz + \int_{L_2} f(z)dz = 2\int_\rho^R \frac{(\ln r)^2}{r^2+1}\,dr - \pi^2 \int_\rho^R \frac{dr}{r^2+1} + 2\pi i \int_\rho^R \frac{\ln r}{r^2+1}\,dr,$$

then,

$$2\int_\rho^R \frac{(\ln r)^2}{r^2+1}\,dr - \pi^2 \int_\rho^R \frac{dr}{r^2+1} + 2\pi i \int_\rho^R \frac{\ln r}{r^2+1}\,dr = -\frac{\pi^3}{4} - \int_{C_\rho} f(z)\,dz - \int_{C_R} f(z)\,dz.$$

Equating real parts on each side of this equation, we have

$$2\int_\rho^R \frac{(\ln r)^2}{r^2+1}\,dr - \pi^2 \int_\rho^R \frac{dr}{r^2+1} = -\frac{\pi^3}{4} - \operatorname{Re}\int_{C_\rho} f(z)\,dz - \operatorname{Re}\int_{C_R} f(z)\,dz;$$

and equating imaginary parts yields

$$2\pi\int_{\rho}^{R}\frac{\ln r}{r^2+1}dr = \mathrm{Im}\int_{C_\rho} f(z)dz - \mathrm{Im}\int_{C_R} f(z)dz.$$

It is straightforward to show that

$$\lim_{\rho\to 0}\int_{C_\rho} f(z)dz = 0 \quad \text{and} \quad \lim_{R\to\infty}\int_{C_R} f(z)dz = 0.$$

Hence

$$2\int_0^\infty \frac{(\ln r)^2}{r^2+1}dr - \pi^2\int_0^\infty \frac{dr}{r^2+1} = -\frac{\pi^3}{4}$$

and

$$2\pi\int_0^\infty \frac{\ln r}{r^2+1}dr = 0.$$

Finally, inasmuch as (see Exercise 1, Sec. 79),

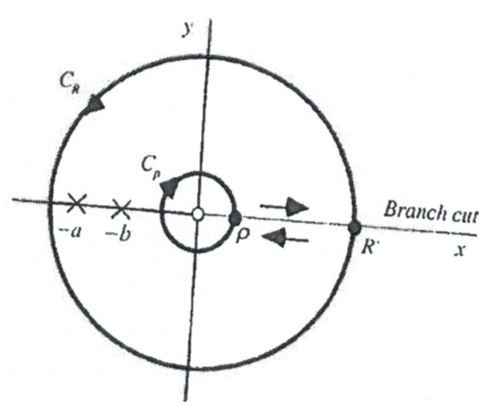
$$\int_0^\infty \frac{dr}{r^2+1} = \frac{\pi}{2},$$

we arrive at the desired integration formulas.

5. Here we evaluate the integral $\displaystyle\int_0^\infty \frac{\sqrt[3]{x}}{(x+a)(x+b)}dx$, where $a > b > 0$. We consider the function

$$f(z) = \frac{z^{1/3}}{(z+a)(z+b)} = \frac{\exp\left(\frac{1}{3}\log z\right)}{(z+a)(z+b)} \qquad (|z| > 0, 0 < \arg z < 2\pi)$$

and the simple closed contour shown below, which is similar to the one used in Sec. 77. The numbers ρ and R are small and large enough, respectively, so that the points $z=-a$ and $z=-b$ are between the circles.



A parametric representation for the upper edge of the branch cut from ρ to R is $z = re^{i0}$ $(\rho \le r \le R)$, and so the value of the integral of f along that edge is

$$\int_\rho^R \frac{\exp\left[\frac{1}{3}(\ln r + i0)\right]}{(r+a)(r+b)}\,dr = \int_\rho^R \frac{\sqrt[3]{r}}{(r+a)(r+b)}\,dr.$$

A representation for the lower edge from ρ to is R is $z = re^{i2\pi}$ $(\rho \le r \le R)$. Hence the value of the integral of f along that edge from R to ρ is

$$-\int_\rho^R \frac{\exp\left[\frac{1}{3}(\ln r + i2\pi)\right]}{(r+a)(r+b)}\,dr = -e^{i2\pi/3}\int_\rho^R \frac{\sqrt[3]{r}}{(r+a)(r+b)}\,dr.$$

According to the residue theorem, then,

$$\int_\rho^R \frac{\sqrt[3]{r}}{(r+a)(r+b)}\,dr + \int_{C_R} f(z)dz - e^{i2\pi/3}\int_\rho^R \frac{\sqrt[3]{r}}{(r+a)(r+b)}\,dr + \int_{C_\rho} f(z)dz = 2\pi i(B_1 + B_2),$$

where

$$B_1 = \operatorname*{Res}_{z=-a} f(z) = \frac{\exp\left[\frac{1}{3}\log(-a)\right]}{-a+b} = -\frac{\exp\left[\frac{1}{3}(\ln a + i\pi)\right]}{a-b} = -\frac{e^{i\pi/3}\sqrt[3]{a}}{a-b}$$

and

$$B_2 = \operatorname*{Res}_{z=-b} f(z) = \frac{\exp\left[\frac{1}{3}\log(-b)\right]}{-b+a} = \frac{\exp\left[\frac{1}{3}(\ln b + i\pi)\right]}{-b+a} = \frac{e^{i\pi/3}\sqrt[3]{b}}{a-b}.$$

Consequently,

$$\left(1 - e^{i2\pi/3}\right)\int_\rho^R \frac{\sqrt[3]{r}}{(r+a)(r+b)}\,dr = -\frac{2\pi i e^{i\pi/3}(\sqrt[3]{a} - \sqrt[3]{b})}{a-b} - \int_{C_\rho} f(z)dz - \int_{C_R} f(z)dz.$$

Now

$$\left|\int_{C_\rho} f(z)dz\right| \le \frac{\sqrt[3]{\rho}}{(a-\rho)(b-\rho)}2\pi\rho = \frac{2\pi\sqrt[3]{\rho}\,\rho}{(a-\rho)(b-\rho)} \to 0 \text{ as } \rho \to 0$$

and

$$\left|\int_{C_R} f(z)dz\right| \le \frac{\sqrt[3]{R}}{(R-a)(R-b)}2\pi R = \frac{2\pi R^2}{(R-a)(R-b)}\cdot\frac{1}{\sqrt[3]{R^2}} \to 0 \text{ as } R \to \infty.$$

Hence

$$\int_0^\infty \frac{\sqrt[3]{r}}{(r+a)(r+b)}dr = -\frac{2\pi i e^{i\pi/3}(\sqrt[3]{a}-\sqrt[3]{b})}{(1-e^{i2\pi/3})(a-b)}\cdot\frac{e^{-i\pi/3}}{e^{-i\pi/3}} = \frac{2\pi i(\sqrt[3]{a}-\sqrt[3]{b})}{(e^{i\pi/3}-e^{-i\pi/3})(a-b)}$$

$$= \frac{\pi(\sqrt[3]{a}-\sqrt[3]{b})}{\sin(\pi/3)(a-b)} = \frac{\pi(\sqrt[3]{a}-\sqrt[3]{b})}{\frac{\sqrt{3}}{2}(a-b)} = \frac{2\pi}{\sqrt{3}}\cdot\frac{\sqrt[3]{a}-\sqrt[3]{b}}{a-b}.$$

Replacing the variable of integration r here by x, we have the desired result:

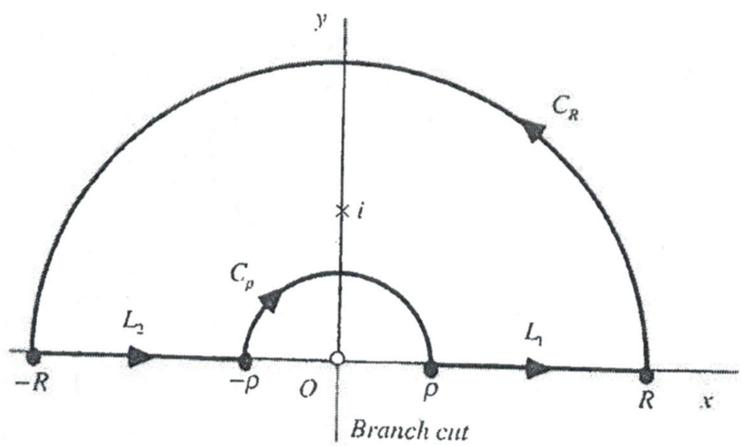
$$\int_0^\infty \frac{\sqrt[3]{x}}{(x+a)(x+b)}dx = \frac{2\pi}{\sqrt{3}}\cdot\frac{\sqrt[3]{a}-\sqrt[3]{b}}{a-b} \qquad (a>b>0).$$

6. (a) Let us first use the branch

$$f(z) = \frac{z^{-1/2}}{z^2+1} = \frac{\exp\left(-\frac{1}{2}\log z\right)}{z^2+1} \qquad \left(|z|>0, -\frac{\pi}{2}<\arg z<\frac{3\pi}{2}\right)$$

and the indented path shown below to evaluate the improper integral

$$\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)}.$$



Cauchy's residue theorem tells us that

$$\int_{L_1} f(z)dz + \int_{C_R} f(z)dz + \int_{L_2} f(z)dz + \int_{C_\rho} f(z)dz = 2\pi i \operatorname*{Res}_{z=i} f(z),$$

or

$$\int_{L_1} f(z)dz + \int_{L_2} f(z)dz = 2\pi i \operatorname*{Res}_{z=i} f(z) - \int_{C_\rho} f(z)dz - \int_{C_R} f(z)dz.$$

Since

$$L_1: z = re^{i0} = r\ (\rho \le r \le R) \quad \text{and} \quad -L_2: z = re^{i\pi} = -r\ (\rho \le r \le R),$$

we may write

$$\int_{L_1} f(z)dz + \int_{L_2} f(z)dz = \int_\rho^R \frac{dr}{\sqrt{r}(r^2+1)} - i\int_\rho^R \frac{dr}{\sqrt{r}(r^2+1)} = (1-i)\int_\rho^R \frac{dr}{\sqrt{r}(r^2+1)}.$$

Thus

$$(1-i)\int_\rho^R \frac{dr}{\sqrt{r}(r^2+1)} = 2\pi i \operatorname*{Res}_{z=i} f(z) - \int_{C_\rho} f(z)dz - \int_{C_R} f(z)dz.$$

Now the point $z = i$ is evidently a simple pole of $f(z)$, with residue

$$\operatorname*{Res}_{z=i} f(z) = \left[\frac{z^{-1/2}}{z+i}\right]_{z=i} = \frac{\exp\left[-\frac{1}{2}\log i\right]}{2i} = \frac{\exp\left[-\frac{1}{2}\left(\ln 1 + i\frac{\pi}{2}\right)\right]}{2i} = \frac{e^{-i\pi/4}}{2i} = \frac{1}{2i}\left(\frac{1-i}{\sqrt{2}}\right).$$

Furthermore,

$$\left|\int_{C_\rho} f(z)dz\right| \le \frac{\pi\rho}{\sqrt{\rho}(1-\rho^2)} = \frac{\pi\sqrt{\rho}}{1-\rho^2} \to 0 \text{ as } \rho \to 0$$

and

$$\left|\int_{C_R} f(z)dz\right| \le \frac{\pi\sqrt{R}}{(R^2-1)} = \frac{\pi}{\sqrt{R}\left(R - \frac{1}{R}\right)} \to 0 \text{ as } R \to \infty.$$

Finally, then, we have

$$(1-i)\int_0^\infty \frac{dr}{\sqrt{r}(r^2+1)} = \frac{\pi(1-i)}{\sqrt{2}},$$

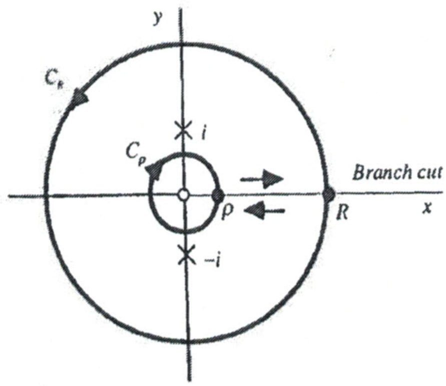
which is the same as

$$\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)} = \frac{\pi}{\sqrt{2}}.$$

(*b*) To evaluate the improper integral $\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)}$, we now use the branch

$$f(z) = \frac{z^{-1/2}}{z^2+1} = \frac{\exp\left(-\frac{1}{2}\log z\right)}{z^2+1} \qquad (|z| > 0, 0 < \arg z < 2\pi)$$

and the simple closed contour shown in the figure below, which is similar to Fig. 103 in Sec. 84. We stipulate that $\rho < 1$ and $R > 1$, so that the singularities $z = \pm i$ are between C_ρ and C_R.



Since a parametric representation for the upper edge of the branch cut from ρ to R is $z = re^{i0}$ $(\rho \le r \le R)$, the value of the integral of f along that edge is

$$\int_\rho^R \frac{\exp\left[-\frac{1}{2}(\ln r + i0)\right]}{r^2+1}\,dr = \int_\rho^R \frac{1}{\sqrt{r}(r^2+1)}\,dr.$$

A representation for the lower edge from ρ to is R is $(\rho \le r \le R)$, and so the value of the integral of f along that edge from R to ρ is

$$-\int_\rho^R \frac{\exp\left[-\frac{1}{2}(\ln r + i2\pi)\right]}{r^2+1}\,dr = -e^{-i\pi}\int_\rho^R \frac{1}{\sqrt{r}(r^2+1)}\,dr = \int_\rho^R \frac{1}{\sqrt{r}(r^2+1)}\,dr.$$

Hence, by the residue theorem,

$$\int_\rho^R \frac{1}{\sqrt{r}(r^2+1)}\,dr + \int_{C_R} f(z)dz + \int_\rho^R \frac{1}{\sqrt{r}(r^2+1)}\,dr + \int_{C_\rho} f(z)dz = 2\pi i(B_1 + B_2),$$

where

$$B_1 = \operatorname*{Res}_{z=i} f(z) = \left[\frac{z^{-1/2}}{z+i}\right]_{z=i} = \frac{\exp\left[-\frac{1}{2}\log i\right]}{2i} = \frac{\exp\left[-\frac{1}{2}\left(\ln 1 + i\frac{\pi}{2}\right)\right]}{2i} = \frac{e^{-i\pi/4}}{2i}$$

and

$$B_2 = \operatorname*{Res}_{z=-i} f(z) = \left[\frac{z^{-1/2}}{z-i}\right]_{z=-i} = \frac{\exp\left[-\frac{1}{2}\log(-i)\right]}{-2i} = \frac{\exp\left[-\frac{1}{2}\left(\ln 1 + i\frac{3\pi}{2}\right)\right]}{-2i} = -\frac{e^{-i3\pi/4}}{2i}.$$

That is,

$$2\int_{\rho}^{R}\frac{1}{\sqrt{r}(r^2+1)}dr=\pi(e^{-i\pi/4}-e^{-i3\pi/4})-\int_{C_\rho}f(z)dz-\int_{C_R}f(z)dz.$$

Since

$$\left|\int_{C_\rho}f(z)dz\right|\le\frac{2\pi\rho}{\sqrt{\rho}(1-\rho^2)}=\frac{2\pi\sqrt{\rho}}{1-\rho^2}\to 0 \text{ as } \rho\to 0$$

and

$$\left|\int_{C_R}f(z)dz\right|\le\frac{2\pi R}{\sqrt{R}(R^2-1)}=\frac{2\pi}{\sqrt{R}\left(R-\frac{1}{R}\right)}\to 0 \text{ as } R\to\infty,$$

we now find that

$$\int_0^\infty\frac{1}{\sqrt{r}(r^2+1)}dr=\pi\frac{e^{-i\pi/4}-e^{-i3\pi/4}}{2}=\pi\frac{e^{-i\pi/4}+e^{-i3\pi/4}e^{i\pi}}{2}$$

$$=\pi\frac{e^{i\pi/4}+e^{-i\pi/4}}{2}=\pi\cos\left(\frac{\pi}{4}\right)=\frac{\pi}{\sqrt{2}}.$$

When x, instead of r, is used as the variable of integration here, we have the desired result:

$$\int_0^\infty\frac{dx}{\sqrt{x}(x^2+1)}=\frac{\pi}{\sqrt{2}}.$$

7

$$B(p,q)=\int_0^1 t^{p-1}(1-t)^{q-1}dt \qquad (p>0,\ q>0$$

$$t=\frac{1}{x+1}\Rightarrow dt=-\frac{dx}{(x+1)^2}$$

$t=1\Rightarrow x=0$

$t=0\Rightarrow x=\infty$

$1-t=1-\frac{1}{x+1}=\frac{x}{x+1}$

$$\Rightarrow B(p,q)=-\int_\infty^0(1+x)^{-(p-1)}\left(\frac{x}{x+1}\right)^{q-1}\frac{dx}{(x+1)^2}$$

$$=\int_0^\infty(1+x)^{-p}x^{q-1}dx$$

$$B(p, 1-p) = \int_0^\infty (1+x)^{-p-(q-p)} x^{q-1} \, dx$$

$q = 1-p > 0$
$\Rightarrow p < 1$
$0 <$

$$= \int_0^\infty \frac{x^{q-1}}{1+x} \, dx = \int_0^\infty \frac{x^{-p}}{x+1} \, dx \, , \quad \text{where } \alpha = q$$

$\Rightarrow$ from example of Sect 84.

$$B(p, 1-p) = \frac{\pi}{\sin(p\pi)} \, , \quad 0 < p < 1$$

14

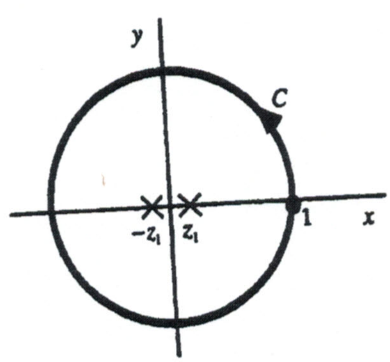
page 290

1 (optional)

Write
$$\int_0^{2\pi}\frac{d\theta}{5+4\sin\theta}=\int_C\frac{1}{5+4\left(\frac{z-z^{-1}}{2i}\right)}\cdot\frac{dz}{iz}=\int_C\frac{dz}{2z^2+5iz-2},$$
where C is the positively oriented unit circle $|z|=1$. The quadratic formula tells us that the singular points of the integrand on the far right here are $z=-i/2$ and $z=-2i$. The point $z=-i/2$ is a simple pole interior to C; and the point $z=-2i$ is exterior to C. Thus
$$\int_0^{2\pi}\frac{d\theta}{5+4\sin\theta}=2\pi i\operatorname*{Res}_{z=-i/2}\left[\frac{1}{2z^2+5iz-2}\right]=2\pi i\left[\frac{1}{4z+5i}\right]_{z=-i/2}=2\pi i\left(\frac{1}{3i}\right)=\frac{2\pi}{3}.$$

2 (optional)

2. To evaluate the definite integral in question, write
$$\int_{-\pi}^{\pi}\frac{d\theta}{1+\sin^2\theta}=\int_C\frac{1}{1+\left(\frac{z-z^{-1}}{2i}\right)^2}\cdot\frac{dz}{iz}=\int_C\frac{4iz\,dz}{z^4-6z^2+1},$$
where C is the positively oriented unit circle $|z|=1$. This circle is shown below.



15

Solving the equation $(z^2)^2-6(z^2)+1=0$ for z^2 with the aid of the quadratic formula, we find that the zeros of the polynomial z^4-6z^2+1 are the numbers z such that $z^2=3\pm 2\sqrt{2}$. Those zeros are, then, $z=\pm\sqrt{3+2\sqrt{2}}$ and $z=\pm\sqrt{3-2\sqrt{2}}$. The first two of these zeros are exterior to the circle, and the second two are inside of it. So the singularities of the integrand in our contour integral are

$$z_1=\sqrt{3-2\sqrt{2}} \quad \text{and} \quad z_2=-z_1,$$

indicated in the figure. This means that

$$\int_{-\pi}^{\pi}\frac{d\theta}{1+\sin^2\theta}=2\pi i(B_1+B_2),$$

where

$$B_1=\operatorname*{Res}_{z=z_1}\frac{4iz}{z^4-6z^2+1}=\frac{4iz_1}{4z_1^3-12z_1}=\frac{i}{z_1^2-3}=\frac{i}{(3-2\sqrt{2})-3}=-\frac{i}{2\sqrt{2}}$$

and

$$B_2=\operatorname*{Res}_{z=-z_1}\frac{4iz}{z^4-6z^2+1}=\frac{-4iz_1}{-4z_1^3+12z_1}=\frac{i}{z_1^2-3}=-\frac{i}{2\sqrt{2}}.$$

Since

$$2\pi i(B_1+B_2)=2\pi i\left(-\frac{i}{\sqrt{2}}\right)=\frac{2\pi}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}=\sqrt{2}\pi,$$

the desired result is

$$\int_{-\pi}^{\pi}\frac{d\theta}{1+\sin^2\theta}=\sqrt{2}\pi.$$

7

Let C be the positively oriented unit circle $|z|=1$. In view of the binomial formula (Sec. 3)

$$\int_0^{\pi}\sin^{2n}\theta\,d\theta=\frac{1}{2}\int_{-\pi}^{\pi}\sin^{2n}\theta\,d\theta=\frac{1}{2}\int_C\left(\frac{z-z^{-1}}{2i}\right)^{2n}\frac{dz}{iz}=\frac{1}{2^{2n+1}(-1)^n i}\int_C\frac{(z-z^{-1})^{2n}}{z}dz$$

$$=\frac{1}{2^{2n+1}(-1)^n i}\int_C\sum_{k=0}^{n}\binom{2n}{k}z^{2n-k}(-z^{-1})^k z^{-1}dz$$

$$=\frac{1}{2^{2n+1}(-1)^n i}\sum_{k=0}^{n}\binom{2n}{k}(-1)^k\int_C z^{2n-2k-1}\,dz.$$

Now each of these last integrals has value zero except when $k = n$:

$$\int_C z^{-1} dz = 2\pi i.$$

Consequently,

$$\int_0^\pi \sin^{2n}\theta \, d\theta = \frac{1}{2^{2n+1}(-1)^n i} \cdot \frac{(2n)!(-1)^n 2\pi i}{(n!)^2} = \frac{(2n)!}{2^{2n}(n!)^2}\pi.$$

problem 1 Evaluate

$$I = \int_0^1 \left(\frac{1-x}{x}\right)^{1/2} \frac{1}{(x+2)^2}\,dx$$

$$I = \int_0^1 h(z)\,R(z)\,dz$$

$$h(z) = \left(\frac{z}{1-z}\right)^{-1/2}, \quad R(z) = \frac{1}{(z+2)^2}$$

For $z \to \infty$ $\quad R(z) = \frac{1}{z^2\left(1+\frac{2}{z}\right)^2} = \frac{1}{z^2} + O\left(\frac{1}{z^3}\right)$

$\Rightarrow \underline{z = \infty}$ is 2nd order zero for $R(z)$

$z \to \infty$: $\quad h(z) = \frac{1}{\left(-1+\frac{1}{z}\right)^{1/2}} = (-1)^{1/2}\left(1 + \frac{1}{2z} + O\left(\frac{1}{z^2}\right)\right)$

$\Rightarrow z = \infty$ is regular (analytical) point for $h(z)$

$\Rightarrow$ $h(z) \cdot R(z)$ has 2nd order zero for $z = \infty$

$$\Rightarrow \operatorname*{Res}_{z=\infty} f(z) = 0$$

We use eq (8) from dumbbelintegration.pdf (attached below) with $\alpha = -\frac{1}{2}$ which gives

$$I = \frac{1}{1-e^{i2\pi(-\frac{1}{2})}} \, 2\pi i \operatorname*{Res}_{z=-2} f(z)$$

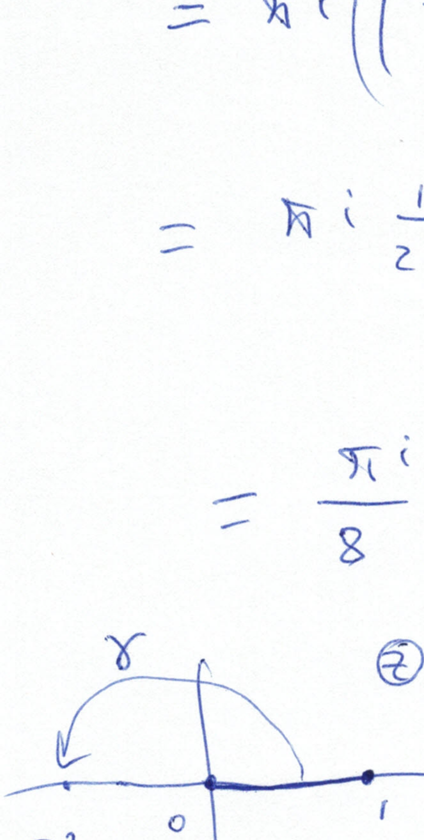
$$= \pi i \operatorname*{Res}_{z=-2} f(z) = \pi i \, h'(-2)$$

$$= \pi i \left(\left(\frac{1}{z} - 1 \right)^{1/2} \right)' \Big|_{z=-2}$$

$$= \pi i \, \frac{1}{2} \, \frac{-\frac{1}{z^2}}{\left(\frac{1}{z}-1\right)^{1/2}} \Big|_{z=-2} = \frac{\pi i}{2} \, \frac{\frac{1}{4}}{\left(-\frac{3}{2}\right)^{1/2}}$$

$$= \frac{\pi i}{8} \sqrt{\frac{2}{3}} \, (\pm i) = \mp \frac{\pi}{4\sqrt{6}}$$

to choose a correct sign we use a principle branch of $h(z)\big|_{z=x+i0} > 0$ for $0<x<1$ and move by contour γ on π which gives $\left(\frac{1}{z}-1\right)^{-1/2} \Rightarrow e^{-\frac{1}{2}\pi i} = -i$



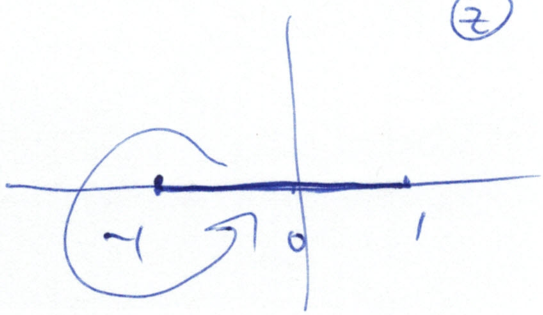
$$\Rightarrow \boxed{I = \frac{\pi}{4\sqrt{6}}}$$

Problem 2

Evaluate

$$I = \int_{-1}^{1} \left[(1-x)(1+x)^3\right]^{1/4} \frac{1}{1+x^2}\, dx$$

Branch cut $[-1,1]$:



Consider a principle branch of

$$h(z) = \left[(1-z)(1+z)^3\right]^{1/4}$$

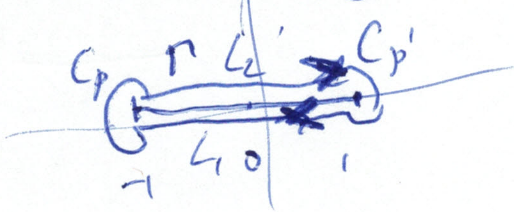
s. t. $h(x+i0) = \left((1-x)(1+x)^3\right)^{1/4} > 0$ for $-1 < x < 1$

then $h(x-i0) = e^{2\pi \cdot 3 \cdot \frac{1}{4} i} h(x) = e^{\frac{3}{2}\pi i} h(x)$ $(*)$

$= -i\, h(x)$

$R(z) := \dfrac{1}{1+z^2}$ and $f(z) := R(z)\, h(z)$

Consider dumbbell contour Γ:



$$\Gamma = L_1 \cup C_\rho \cup L_2 \cup C_{\rho'}$$

Similar to dumblelintegration.pdf,

$$\int_{C_\rho} f(z)\,dz = \int_{C_{\rho'}} f(z)\,dz = 0 \quad \text{as } \rho \to 0$$

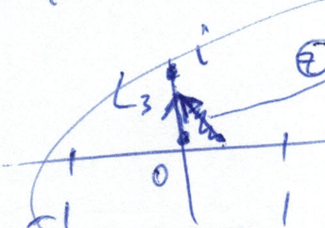
$\Rightarrow$ as $\rho \to 0$

$$\int_{-1}^{1} f(x)\,dx + \int_{1}^{-1} \underset{-i}{\underset{\|}{e^{\frac{3\pi i}{2}}}} f(x)\,dx$$

$$= (1+i)\int_{-1}^{1} f(x)\,dx = (1+i)I$$

$$= 2\pi i \left(\operatorname*{Res}_{z=i} f(z) + \operatorname*{Res}_{z=-i} f(z) + \operatorname*{Res}_{z=\infty} f(z) \right) \quad (**)$$

here $\operatorname*{Res}_{z=i} f(z) = \left.\frac{h(z)}{2z}\right|_{z=i} = e^{\frac{i}{4}\left(-\frac{\pi}{4} + 3\cdot\frac{\pi}{4}\right)} \left.\frac{h(i)}{2i}\right| =$



change of arg $(1-z)$ over

L_3 : $\mathrm{Arg}(1-i) = -\frac{\pi}{4}$

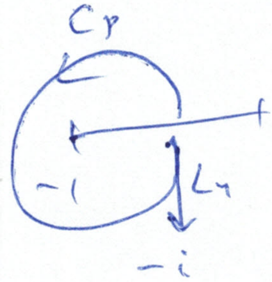
$\mathrm{Arg}(1+i) = \frac{\pi}{4}$ - for arg $(z+1)$

$$= e^{i\frac{\pi}{8}} \frac{\sqrt{2}}{2i} = -\frac{i}{\sqrt{2}} e^{i\frac{\pi}{8}} \quad (***)$$

Similar following C_ρ and L_4:

$$h(-i) = \sqrt{2}\, e^{\frac{i}{4}\left(\frac{\pi}{4} - 3\frac{\pi}{4}\right) + i\frac{3}{2}\pi i}$$

from (*)



$$= \sqrt{2}\, e^{i\pi\frac{11}{8}}$$

then $$\operatorname*{Res}_{z=-i} f(z) = \frac{h(z)}{2z}\Big|_{z=-i} = \frac{i}{\sqrt{2}}\, e^{i\pi\frac{11}{8}} \quad (****)$$

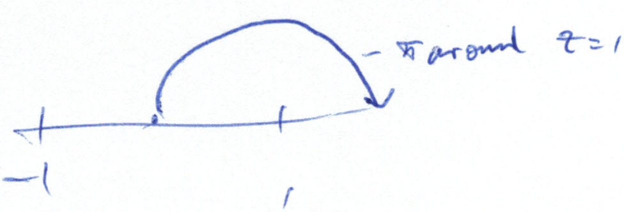
$\operatorname*{Res}_{z=\infty} f(z)$: For $z \to \infty$

$$f(z) = \left[(1-z)(1+z)^3\right]^{1/4} \frac{1}{1+z^2}$$

$$= \frac{\left(-z^4\left(1-\frac{1}{z}\right)\left(1+\frac{1}{z}\right)\right)^{1/4}}{1+z^2} = e^{-i\frac{\pi}{4}}\left(\frac{1}{z} + \frac{1}{2z^2} + \dots\right)$$

$$\Downarrow$$

$$\operatorname*{Res}_{z=\infty} f(z) = -e^{-i\frac{\pi}{4}} \quad (*****)$$



From $(**) - (*****)$:

$$I = \frac{2\pi i}{1+i}\left(\operatorname*{Res}_{z=i} f + \operatorname*{Res}_{z=-i} f + \operatorname*{Res}_{z=\infty} f\right)$$

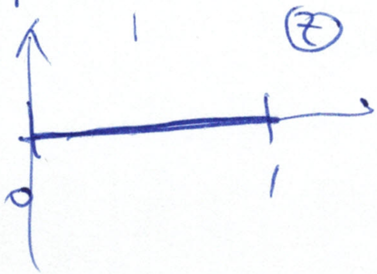
$$= \frac{2\pi i}{1+i}\left(-\frac{i}{\sqrt{2}}\, e^{i\frac{\pi}{8}} + \frac{i}{\sqrt{2}}\, e^{i\pi\frac{11}{8}} - e^{-i\frac{\pi}{4}}\right)$$

$$\boxed{= \pi\sqrt{2}\left(\sqrt{2}\cos\frac{\pi}{8} - 1\right) = I}$$

Problem 3

$$I=\int_0^1 \left(x(1-x)^3\right)^{1/4}\frac{1}{(1+x)^3}\,dx$$

Branch cut $[0,1]$



$$h(z)=\left(z(1-z)^3\right)^{1/4}$$

$$R(z)=\frac{1}{(1+z)^3}$$

principle branch: $h(x+i0)=h(x)>0$ for $0<x<1$

$$\Rightarrow h(x-i0)=e^{i2\pi\frac{1}{4}}h(x)=i\,h(x)$$

For dumbbell contour for $\rho\to 0$ we obtain as in dumbbellintegral.pdf:

$$\int_0^1 f(x)+\int_1^0 f(x-i0)\,dx=(1-i)I$$

$$=2\pi i\left(\operatorname*{Res}_{z=-1}f(z)+\operatorname*{Res}_{z=\infty}f(z)\right)$$

$z=\infty$ is 2nd order zero

for $h(z)\ R(z) = \frac{z}{(1+z)^3}\left(\frac{1}{z}-1\right)^{3/4}$

$$\Rightarrow \operatorname*{Res}_{z=\infty} f(z) = 0$$

$z=-1$ is the pole of order 3,

then $\operatorname*{Res}_{z=-1} f(z) = \frac{1}{2!} h''(-1) =$

$$= \left(-\frac{3}{2}\right) 8^{-7/4} e^{i\frac{\pi}{4}}$$

$$\Rightarrow I = \frac{2\pi i}{1-i} \operatorname*{Res}_{z=-1} f(z)$$

$$= \frac{2\pi i}{\sqrt{2}\, e^{-i\frac{\pi}{4}}} \left(-\frac{3}{2}\right) 8^{-7/4} e^{i\frac{\pi}{4}}$$

$$= \frac{3\pi\, 2^{1/4}}{64}$$

Evaluation of definite integrals from dumbbell contours

Pavel M. Lushnikov
Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico, USA
(Dated: Fall 2024)

We consider the definite integrals of the β-function type

$$I = \int_0^1 \left(\frac{x}{1-x}\right)^\alpha R(x)dx, \qquad -1 < \alpha < 1, \tag{1}$$

where $R(x)$ is the rational function such that it does not have poles at the closed interval $x \in [0,1]$ and

$$R(x) \to const \text{ for } x \to \infty. \tag{2}$$

To evaluate (1) we extend its integrand into the complex plane $z \in \mathbb{C}$ as follows

$$f(z) := \left(\frac{z}{1-z}\right)^\alpha R(z) \tag{3}$$

and define a branch cut at the segment of the real line $[0,1]$ which connects branch point $z=0$ and $z=1$ of $f(z)$. Then we choose the branch of $f(z)$ such that

$$f(x+i0) = f(x) > 0 \qquad \text{for} \qquad 0 < x < 1. \tag{4}$$

Here and below $x+i0$ and $x-i0$ means the limit $\epsilon \to 0^+$, $\epsilon > 0$ for $x+i\epsilon$ and $x-i\epsilon$, respectively.

To obtain $f(x-i0),\ 0<x<1$ we move from $x+i0,\ 0<x<1$ to $x-i0,\ 0<x<1$ either around the branch point $z=0$ in the counterclockwise (positive) direction on the angle 2π thus adding $2\pi\alpha$ to the argument of $f(z)$ from z^α factor in (3) or around the branch point $z=1$ in the clockwise (negative) direction on the angle -2π thus adding $-2\pi(-\alpha) = 2\pi\alpha$ to the argument of $f(z)$ from $(1-z)^{-\alpha}$ factor in (3). Thus in both cases

$$f(x-i0) = e^{i2\pi\alpha} f(x+i0) \qquad \text{for} \qquad 0 < x < 1. \tag{5}$$

It also proves that $f(z)$ is analytic in $\mathbb{C} \setminus [0,1]$.

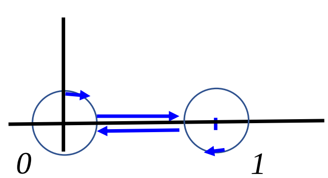


FIG. 1. Dumbbell contour.

We integrate over a dumbbell contour shown in Fig. 1 consisting of the line segments $L_1:\ [1-\rho-i0, \rho-i0]$, $L_2:\ [\rho+i0, \rho+i0]$ and the circles $C_\rho:\ |z|=1$, $C'_\rho:\ |1-z|=1$ with $0<\rho\ll 1$. Here ρ is chosen small enough such that now poles of $R(z)$ are in interior or on of any of these two circles. It implies that all poles of $R(z)$ are exterior to to the dumbbell contour $\Gamma_\rho := L_1 \cup C_\rho \cup L_2 \cup C'_\rho$. Then the residue theorem implies that

$$I_\rho := \int\limits_{\Gamma_\rho} f(z)dz = 2\pi i \left[\sum_{k=1}^{n} Res_{z=z_k} f(z) + Res_{z=\infty} f(z) \right], \tag{6}$$

where $z_1, \dots, z_n$ are the residues of $f(z)$ for $z \in \mathbb{C}$

The definition of Γ_ρ and (5) also imply that

$$\begin{aligned} I_\rho &= \int\limits_{\Gamma_\rho} f(z)dz = \int\limits_{L_1} f(z)dz + \int\limits_{C_\rho} f(z)dz + \int\limits_{L_2} f(z)dz + \int\limits_{C'_\rho} f(z)dz \\ &= \int\limits_{1-\rho-i0}^{\rho-i0} f(z)dz + \int\limits_{C_\rho} f(z)dz + \int\limits_{\rho+i0}^{1-\rho+i0} f(z)dz + \int\limits_{C'_\rho} f(z)dz \\ &= e^{i2\pi\alpha} \int\limits_{1-\rho+i0}^{\rho+i0} f(z)dz + \int\limits_{C_\rho} f(z)dz + \int\limits_{\rho+i0}^{1-\rho+i0} f(z)dz + \int\limits_{C'_\rho} f(z)dz. \end{aligned} \tag{7}$$

We prove that $\lim\limits_{\rho\to 0} \int\limits_{C_\rho} f(z)dz = 0$ as follows:

$$\left| \int\limits_{C_\rho} f(z)dz = 0 \right| \leq \int\limits_{|C_\rho|} |f(z)||dz| \leq \frac{M_1 \rho^\alpha}{(1-\rho)^\alpha} \int\limits_{|C_\rho|} |dz| = 2\pi \frac{M_1 \rho^\alpha}{(1-\rho)^{\alpha+1}} \to 0 \text{ as } \rho \to 0^+$$

because $-1 < \alpha < 1$. Here $M_1 = \max\limits_{C_\rho} |R(z)|$ and $|C_\rho|$ means that the integral is taken is the positive direction. In a similar way we prove that $\lim\limits_{\rho\to 0} \int\limits_{C'_\rho} f(z)dz = 0$.

Thus taking the limit $\rho \to 0^+$ in (7) and using (6) we obtain that

$$I = \frac{1}{1 - e^{i2\pi\alpha}} 2\pi i \left[\sum_{k=1}^{n} Res_{z=z_k} f(z) + Res_{z=\infty} f(z) \right]. \tag{8}$$

To find $Res_{z=\infty} f(z)$ we consider the Laurent series of $f(z)$ at $z = \infty$ by first finding Laurent series for $R(z)$ and $q(z) := \left(\frac{z}{1-z}\right)^\alpha$. For $R(z)$ we use (2) to obtain the Laurent series

$$R(z) = c_0 + \frac{c_{-1}}{z} + \frac{c_{-2}}{z^2} + \dots, \; |z| > R_0, \tag{9}$$

where $R_0 > 0$ is chosen to be large nought such that all finite poles of $R(z)$ are located in $|z| < R_0$.

For $q(z)$ we obtain that

$$q(z) = \left(\frac{z}{1-z}\right)^\alpha = \left(-\frac{1}{1-\frac{1}{z}}\right)^\alpha = e^{i\alpha\pi} \left(\frac{1}{1-\frac{1}{z}}\right)^\alpha = e^{i\alpha\pi} \left[1 + \frac{\alpha}{z} + \dots\right], \; |z| > 1, \tag{10}$$

where we used the Taylor series for $w := \frac{1}{z}$ and we moved from $z = x + i0,\ 0 < x < 1$ to $z = x \gg 1$ by moving around the branch point $z = 1$ in the negative direction on the argument $-\pi$ around the branch point $(1-z)^{-alpha}$ thus accumulating an addition to the argument of $q(z)$ as $(-\pi)(-\alpha) = \pi\alpha$ thus giving the factor $e^{i\alpha\pi}$.

Combining (9) and (10) we obtain the Laurent series for $f(z)$ as

$$f(z) = R(z)q(z) = e^{i\alpha\pi} \left[c_0 + \frac{\alpha c_0 + c_{-1}}{z} + \dots\right], \; |z| > R_0, \tag{11}$$

which gives that

$$Res_{z=\infty} f(z) = -e^{i\alpha\pi} (\alpha c_0 + c_{-1}). \tag{12}$$

Together with (8) and (12) we thus evaluate the definite integral (1).