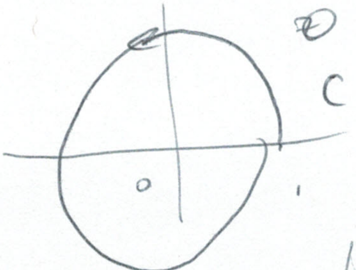
math 561   HW 10 Solutions

p. 296

①

$C : |z| = 1$



(a) $f(z) = z^2$

By thm of Sec 86:

(a) f meromorphic in interior of C

(b) f is analytic and nonzero on C

(c) Number of zeros inside C

Counting multiplicity: $Z = 2$

Number of poles inside C: $P = 0$

$\Rightarrow \Delta_C \arg f(z) = 2\pi(Z - P) = 4$

(b) $f(z) = \dfrac{z^3+2}{z}$ — 1 simple pole at $z = 1$, $P = 1$

$z^3 = -2 \Rightarrow |z| = 2^{1/3}$, i.e. no zeros inside C, $Z = 0$

$\Rightarrow \Delta_C \arg f(z) = 2\pi(Z - P) = -2\pi$

(c) $f(z) = \frac{(2z-1)^7}{z^3}$

1 rd order pole at $z=0 \Rightarrow P = 3$

1 zero of multiplicity ~~2~~ 7 at $z=0$

$\Rightarrow Z = 7$

$\Delta_C \arg f(z) = 2\pi(Z-P) = 2\pi(7-3)$

$= 8\pi$

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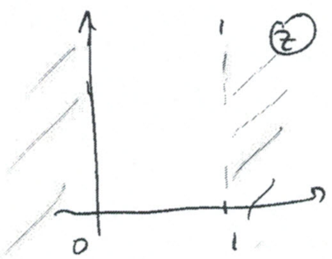
$w = u + iv$

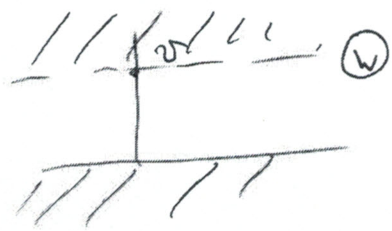
(1) $w = iz = e^{i\frac{\pi}{2}} z$

$z = re^{i\varphi} \Rightarrow w = re^{i\left(\varphi + \frac{\pi}{2}\right)} \Rightarrow$ rotation at angle $\frac{\pi}{2}$



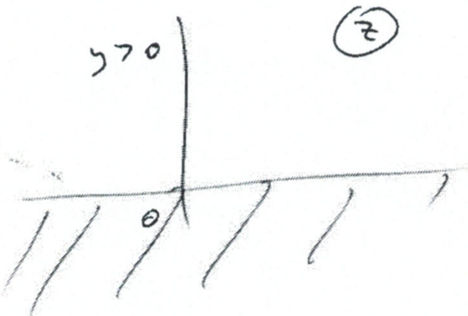
$\rightarrow$ $x = 0$ maps into $-\infty < u < \infty$, $v = 0$

$x = \pi$ maps into $u + i$, $-\infty < u < \infty$

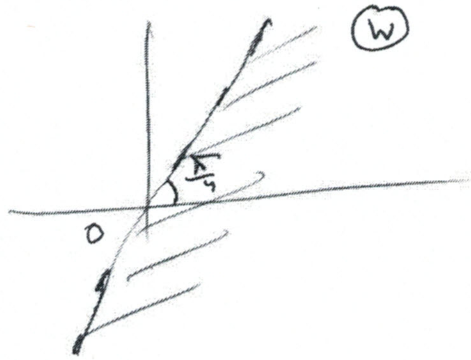


i.e. strip $0 < v < 1$

(3) $w = (1+i)z = \sqrt{2}\, e^{i\frac{\pi}{4}} z \Rightarrow$ stretching by $\sqrt{2}$ and rotation by $\frac{\pi}{4}$



$\rightarrow$



$\Rightarrow v > u$

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1 Linear fractional transformation

mapping $z_1 = 2,\ z_2 = i,\ z_3 = -2$

onto $w_1 = 1,\ w_2 = i,\ w_3 = -1$

use eq (1) of section 94:

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(w-1)(i+1)}{(w+1)(i-1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$= \frac{(z-2)(i+2)}{(z+2)(i-2)}$$

Bring to common denominator and solve for w:

$$(w-1)(i+1)(z+2)(i-2) = \cancel{(z-2)(i+2)}(z-2)(i+2)(w+1)(i-1)$$

$$w\left[(i+1)(z+2)(i-2) - (z-2)(i+2)(i-1)\right] = (i+1)(z+2)(i-2)$$

$$+(z-2)(i+2)(i-1)$$

$$\Rightarrow w = \left\{\frac{z\left[(i+1)(i-2)-(i+2)(i-1)\right] + 2(i+1)(i-2) + 2(i+2)(i-1)}{z\left[(i+1)(i-2)+(i+2)(i-1)\right] + 2(i+1)(i-2) - 2(i+2)(i-1)}\right\}$$

$$= \frac{-6z-4i}{-2iz+12} = \boxed{\frac{3z+2i}{iz+6} = w}$$

$$\left\{\begin{aligned}(i+1)(i-2) &= -i-3\\ (i-1)(i+2) &= i-3\end{aligned}\right\}$$

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(2)

(c) $z_1 = -i \qquad z_2 = 0 \qquad z_3 = i$

$\cancel{w}_1 = -1 \qquad w_2 = i \qquad w_3 = 1$

use eq (1) of Sec (9.4) $\Rightarrow$

$$\frac{(w+1)(i-1)}{(w-1)(i+1)} = \frac{(z+i)(-i)}{(z-i)\,\cancel{i}}$$

$$\frac{(i-1)(1-i)}{1+1} = i$$

$$\frac{w+1}{w-1}\,i = -\frac{(z+i)}{z-i} \Rightarrow i(w+1)(z-i) = -(z+i)(w-1)$$

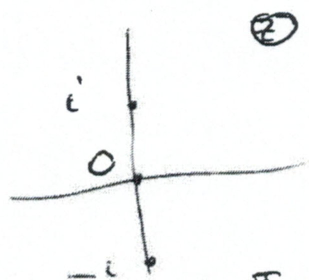
Collect terms with w:

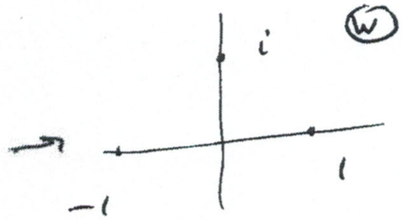
$$w\left[i\cancel{z}+1+(z+i)\right] = -i(z-i)+(z+i) = z(1-i)+(i-1)$$

$$w\left[z(i+1)+(i+1)\right]$$

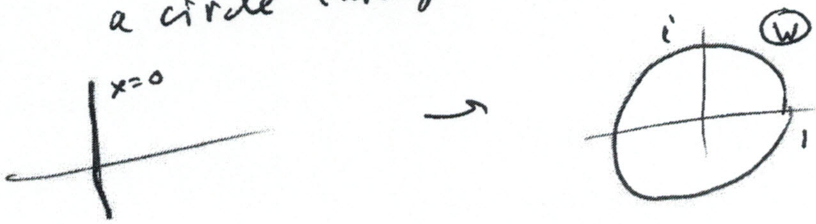
$$\Rightarrow \boxed{w = \frac{z(1-i)+(i-1)}{(z+1)\,i+1}} = \frac{(z-1)}{z+1}\frac{(i-1)}{(i+1)} \boxed{= -i\,\frac{z-1}{z+1}}$$

$$\frac{(i-1)(1-i)}{1+1} = -i$$





I.e. line through i, 0, i in z transforms into a circle through -1, i, 1 in w, i.e.



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③ ③

$z_1 = \infty \quad z_2 = i \quad z_3 = 0$

$w_1 = 0 \quad w_2 = i \quad w_3 = \infty$

$$\lim_{w_3 \to \infty} \frac{w(i - w_3)}{(w - w_3) i} = \frac{w}{i} = \lim_{z_1 \to \infty} \frac{(z - z_1)(i - 0)}{(z - 0)(i - z_1)} = \frac{i}{z}$$

$$\Rightarrow \boxed{w = -\frac{1}{z}}$$

Similar to class:

⑤ $Z = T(z) = \frac{a_1 z + b_1}{c_1 z + d_1}$ (*), $(a_1 d_1 - b_1 c_1 \neq 0)$

$$w = S(Z) = \frac{a_2 Z + b_1}{c_1 Z + d_1} \qquad ((a_2 d_2 - b_2 c_2 \neq 0)$$

$$= \{\text{plug in } (*)\} = \frac{a_2 \frac{a_1 z + b_1}{c_1 z + d_1} + b_2}{c_2 \frac{a_1 z + b_1}{c_1 z + d_1} + d_2}$$

$$= \frac{a_2(a_1 z + b) + b_2(c_1 z + d_1)}{c_2(a_1 z + b_1) + d_2(c_1 z + d_1)} = \frac{z(a_1 a_2 + b_2 c_1) + a_2 b_1 + b_2 d_1}{z(a_1 c_2 + c_1 d_2) + b_1 c_2 + d_1 d_2}$$

$$= \frac{a_3 z + b_3}{c_3 z + d_3}, \quad \text{where } \begin{aligned} a_3 &= a_1 a_2 + b_2 c_1 \\ b_3 &= a_2 b_1 + b_2 d_1 \\ c_3 &= a_1 c_2 + c_1 d_2 \\ d_3 &= b_1 c_2 + d_1 d_2 \end{aligned}$$

and $a_3 d_3 - b_3 c_3 = (a_1 a_2 + b_2 c_1)(b_1 c_2 + d_1 d_2) - (a_2 b_1 + b_2 d_1)$
$\times (b_1 c_2 + d_1 d_2) = (a_1 d_1 - b_1 c_1)(a_2 d_2 - b_2 c_2) \neq 0$

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(3)

(a) $(*)\ w = \frac{i-z}{i+z} \Rightarrow$ solve for z:

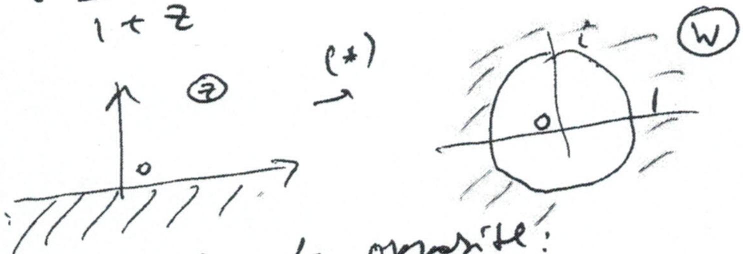
$$(z+i)w = i-z$$

$$z(w+1) = w(-i)+i \Rightarrow z = \frac{i(1-w)}{1+w}$$

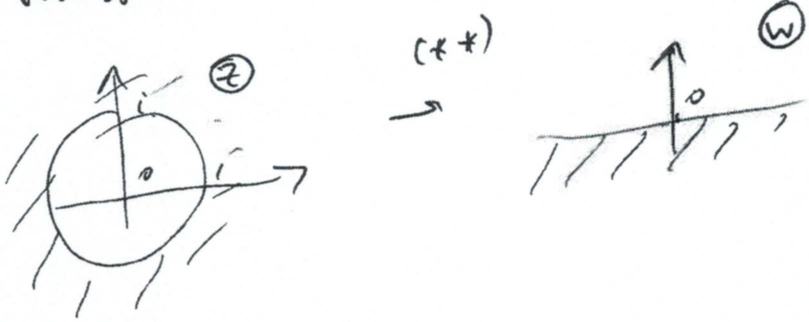
$\Rightarrow$ inverse transformation ($z \leftrightarrow w$):

$$w = \frac{i(1-z)}{1+z} \quad (**)$$

Appendix 2, Fig 13: (*)



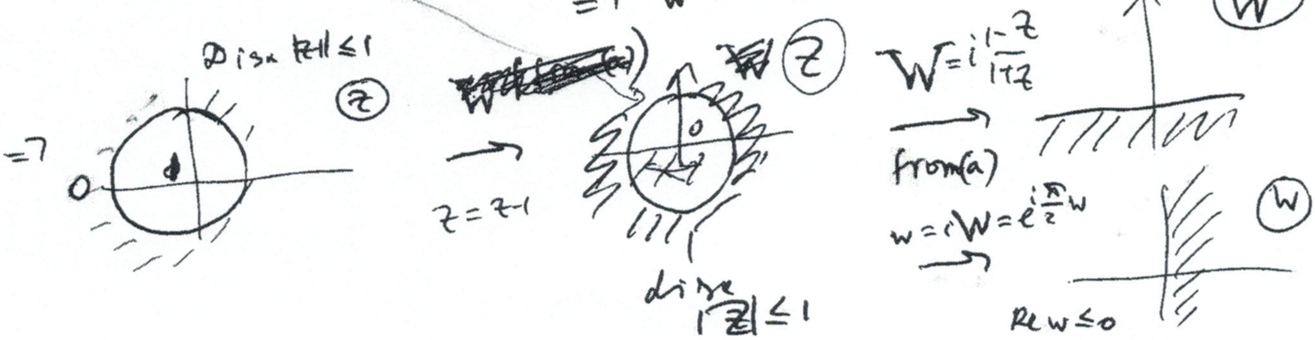
$\Rightarrow$ inverse transform (**) do opposite:



(b) $w = \frac{z-2}{z}$

$Z = z-1$, $W = i\frac{1-Z}{1+Z} = i\frac{1-(z-1)}{1+(z-1)} = i\frac{-z+2}{z} = -iw$

$\Rightarrow w = iW$



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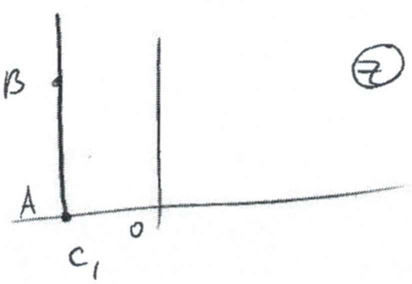
1. $w = \sin z = u + iv$

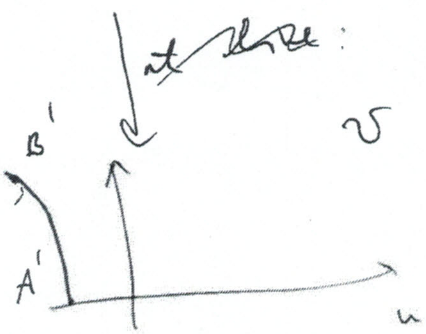
$z = c_1 + iy \quad , -\frac{\pi}{2} < c_1 < 0 \quad , y > 0 \Rightarrow c_1 = -|c_1|$



$\sin z = \sin x \cosh y + i \cos x \sinh y$

$\Rightarrow u = \sin x \cosh y$
$\quad v = \cos x \sinh y$
$\Bigg\}$ (1) $\Rightarrow \frac{u^2}{\cos^2(x)} - \frac{v^2}{\sin^2(x)} = 1$

hyperbola.



$u = \underbrace{\sin(-|c_1|)}_{<0} \underbrace{\cosh y}_{>0}$

$v = \underbrace{\cos(-|c_1|)}_{>0} \underbrace{\sinh y}_{>0}$

$\Rightarrow$ curve lies in upper left quadrant

A': $u = -\sin|c_1|$
$\quad v = 0$

B' $\quad u = -\sin|c_1| \cosh y < 0$
$\quad v = \cos|c_1| \sinh y > 0$

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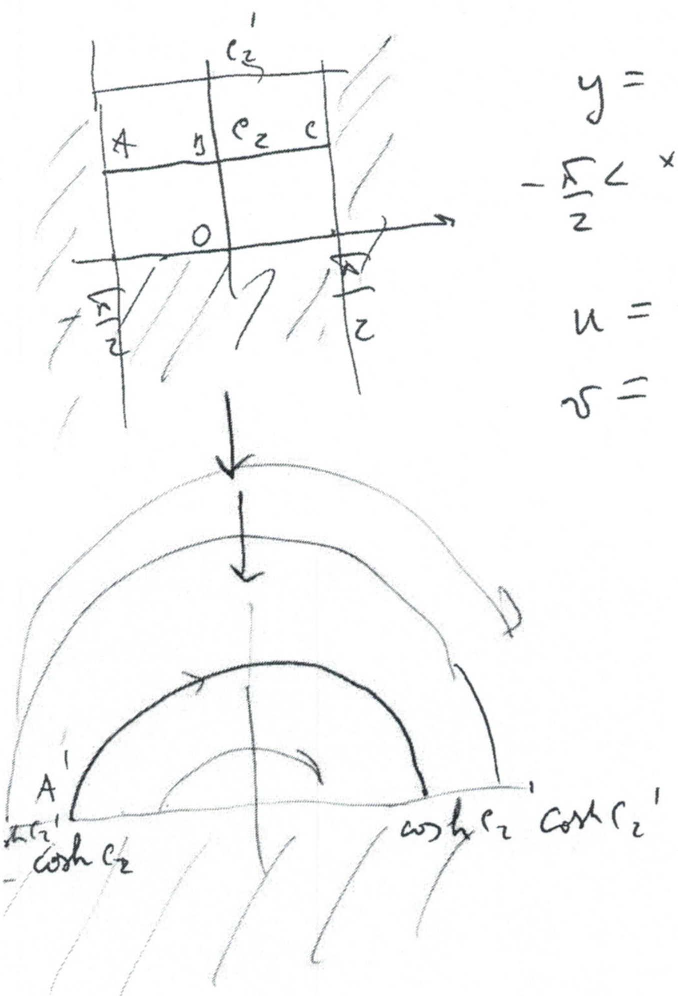
(3) $w = \sin z$ (2)



$y = c_2 > 0$

$-\frac{\pi}{2} < x < \frac{\pi}{2}$

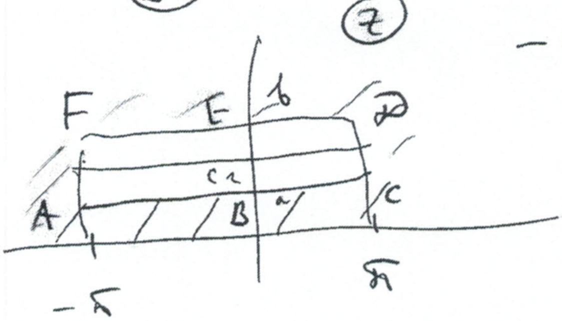
$u = \sin x \cosh c_2$ > 0

$v = \cos x \sinh c_2$

> 0

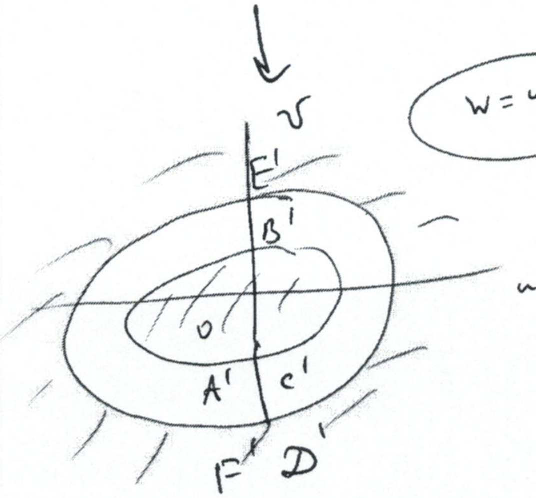
$\Rightarrow$ maps into $v > 0$.

5



$-\pi \le x \le \pi$

$a \le y \le b$



$w = u + iv$

For each fixed $y = c$ we obtain ellips:

$$\left.\begin{aligned} u &= \sin x \cosh c_2 > 0 \\ v &= \cos x \sinh c_2 \end{aligned}\right\}$$

> 0

— parametric form of ellips in (u, v) plane with $-\pi \le x \le \pi$

$$\left.\begin{aligned} F' &= D' \\ A' &= C' \end{aligned}\right\} \Rightarrow \text{mapping not one to one.}$$

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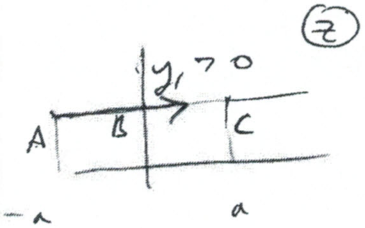
1

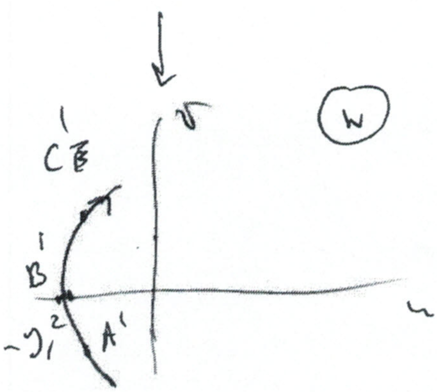
① $w = f(z) = z^2 = x^2 - y^2 + 2ixy = u + iv$

$z = re^{i\theta}$, $w = Re^{i\phi}$

$\Rightarrow w = Re^{i\phi} = (re^{i\theta})^2 = r^2 e^{2i\theta}$

$\Rightarrow R = r^2$, $\phi = 2\theta$





$u = x^2 - y^2$, $v = 2xy$

$y = y_1 = \text{const}$

$\Rightarrow \left.\begin{array}{l} u = x^2 - y_1^2 \\ v = 2xy_1 \end{array}\right\} \Rightarrow x = \dfrac{v}{2y_1}$

$\Rightarrow u = \dfrac{v^2}{4y_1^2} - y_1^2$

A + B: $x = 0$, $y = y_1$

$\Rightarrow u = -y_1^2$

$v = 0$

$\Rightarrow v^2 = 4y_1^2(u + y_1^2)$

parabola with focus at $w = 0$

A': A $x = -a$, $y = y_1$

$u = a^2 - y_1^2$

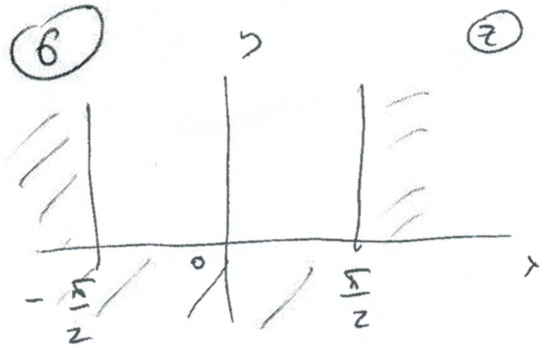
$v = -2ay_1$

~~A~~ B: $x = a$, $y = y_1$

$u = a^2 - y_1^2$

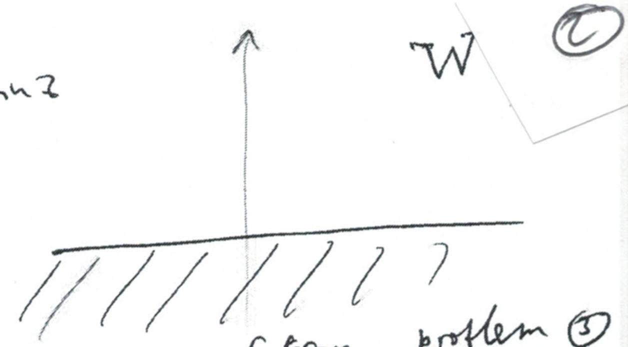
$v = 2ay_1$

⑥



$W = \sin z$

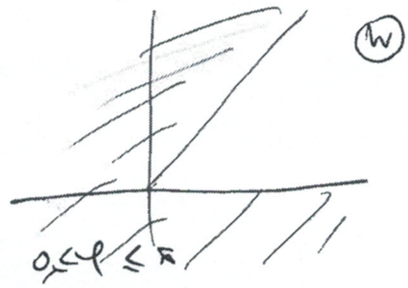
$\longrightarrow$



(from problem ⑤ page 33)

$w = W^{1/4}$

$\longrightarrow$



$W = Re^{i\varphi} \quad 0 \le \varphi \le \pi$

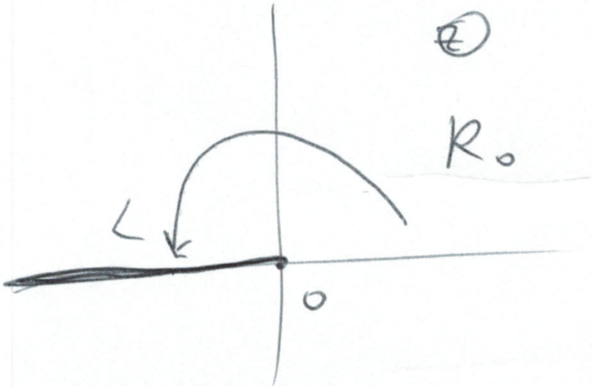
$\Rightarrow$ for principle branch

$w = R^{1/4} e^{i\varphi/4} \Rightarrow 0 \le \frac{\varphi}{4} \le \frac{\pi}{4}$

page 350 (1)

$w = \log z$

branch cut at $-\infty < x \leq 0$

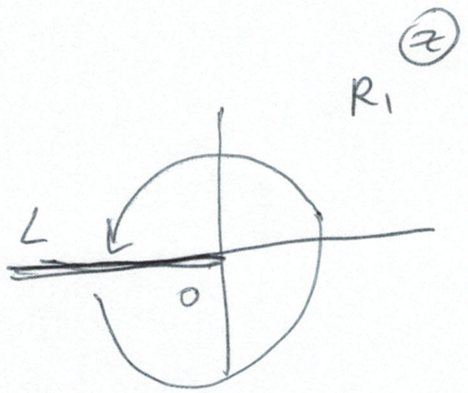


at sheet R_0: $-\pi < \mathrm{Im}(\log z) < \pi$

at $0 < x < \infty$,

$\log z = \ln x \in \mathbb{R}$

$-\pi < \theta < \pi$



$+\pi < \mathrm{Im}(\log z) < 3\pi$

upper edge of ~~R_0~~ L at R_0

is joined to lower edge of L at R_1

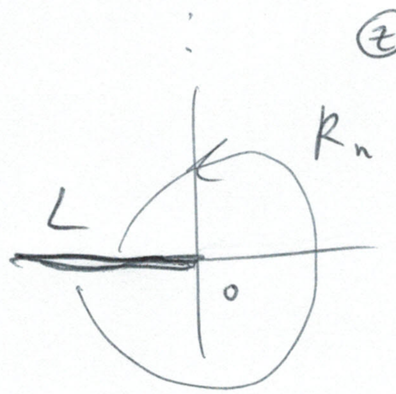
⋮

$\pi < \theta < 3\pi$

upper edge of L at R_n

is joined to lower edge of L

at R_{n+1}

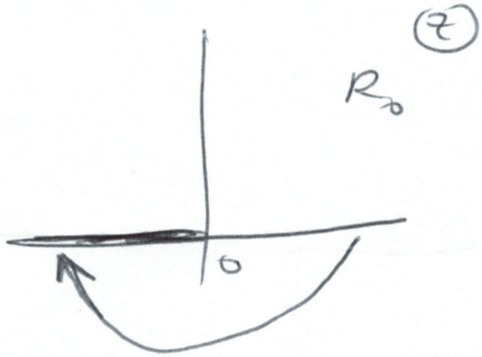


$(2n-1)\pi < \mathrm{Im}(\log z) < (2n+1)\pi$

$(2n-1)\pi < \theta < (2n+1)\pi$

⋮

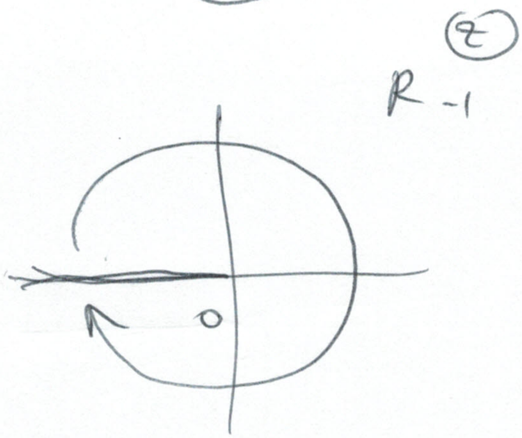
R_0 rotating in opposite direction produces sheets $R_{-1}, R_{-2}, \ldots$



sheet R_0:

$$-\pi < \mathrm{Im}(\log z) < \pi$$

$$-\pi < \theta < \pi$$

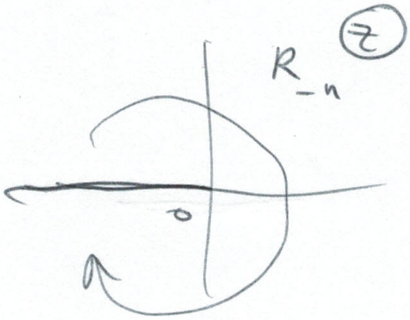


sheet R_{-1}:

$$-3\pi < \mathrm{Im}(\log z) < -\pi$$

$$-3\pi < \theta < -\pi$$

⋮



sheet R_{-n}:

$$-(2n+1)\pi < \mathrm{Im}(\log z) < -(2n-1)\pi$$

⋮

(1) Lower edge of L at R_0

is joined to upper edge of R_{-1}

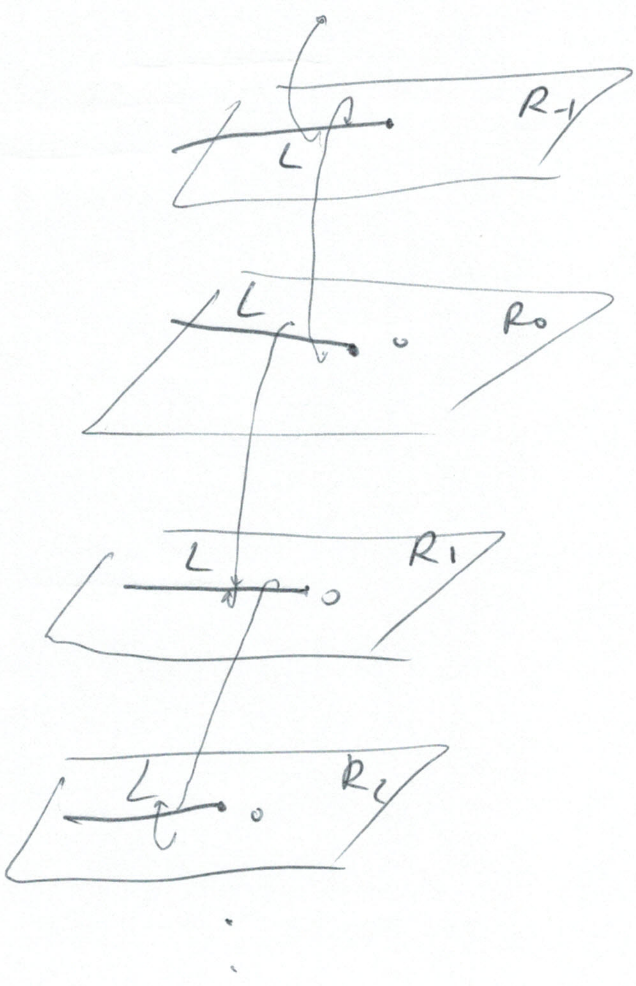
(2) Lower edge of L at R_{-1}

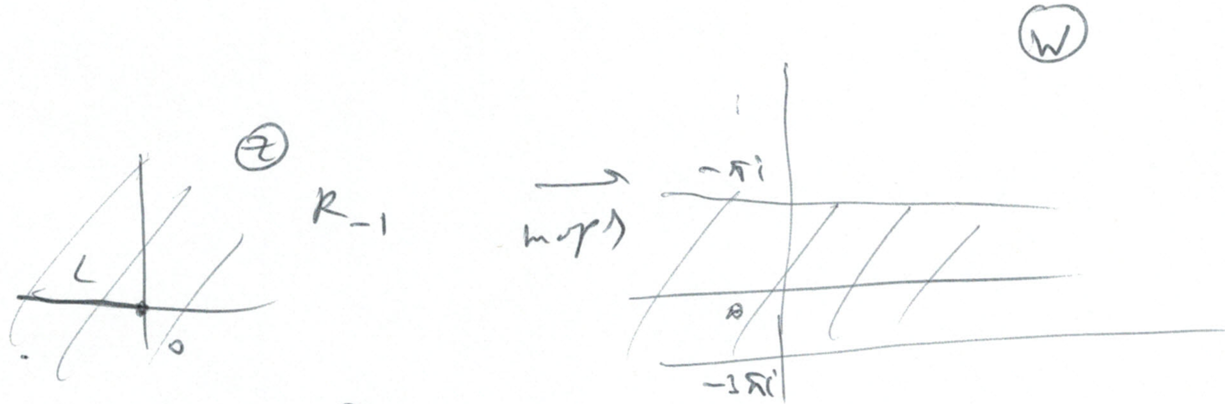
is ~~to~~ joined to upper edge of ~~L at~~ R_{-2}

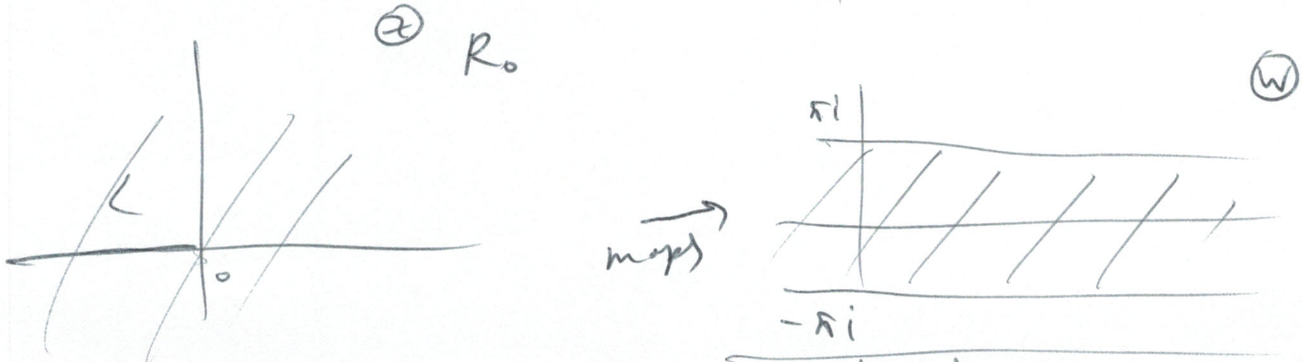
⋮

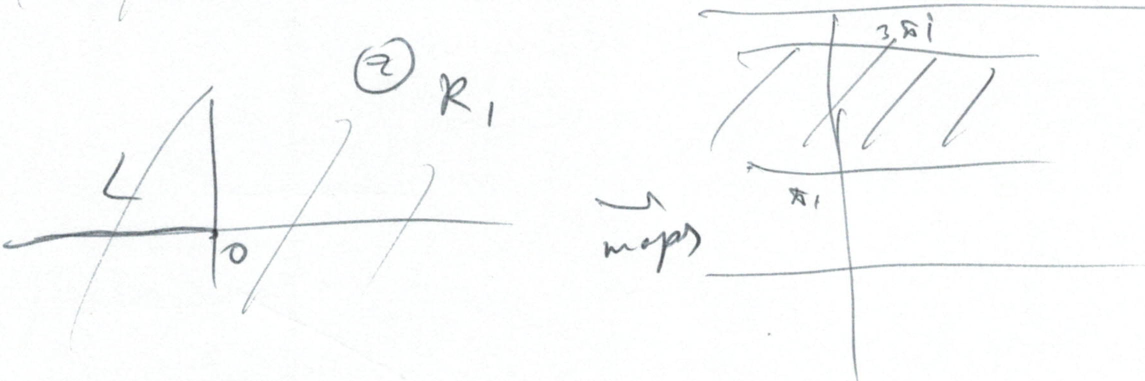
(n+1) Lower edge of L at R_{-n}

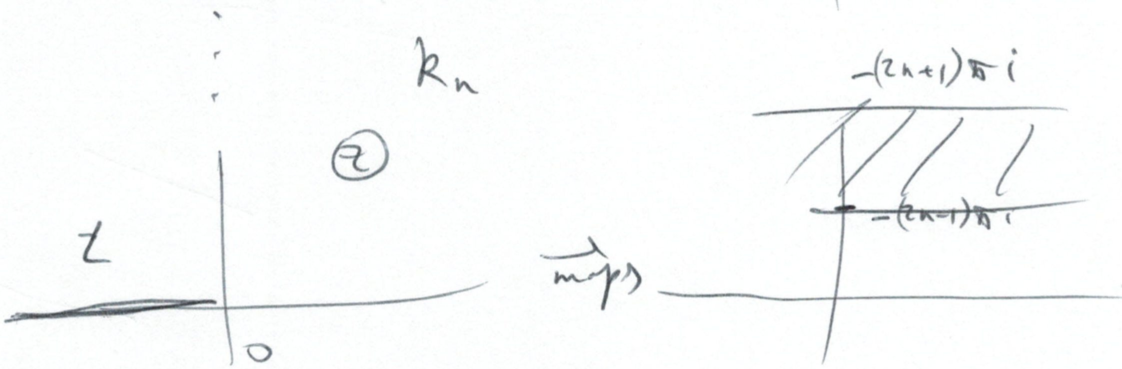
is joined to upper edge of L at R_{-n-1}











Combining together: all sheets $R_{-\infty}, \dots R_0, R_1 \dots R_\infty$ are mapped one-to-one to entire plane $\mathbb{C}$ in (w).

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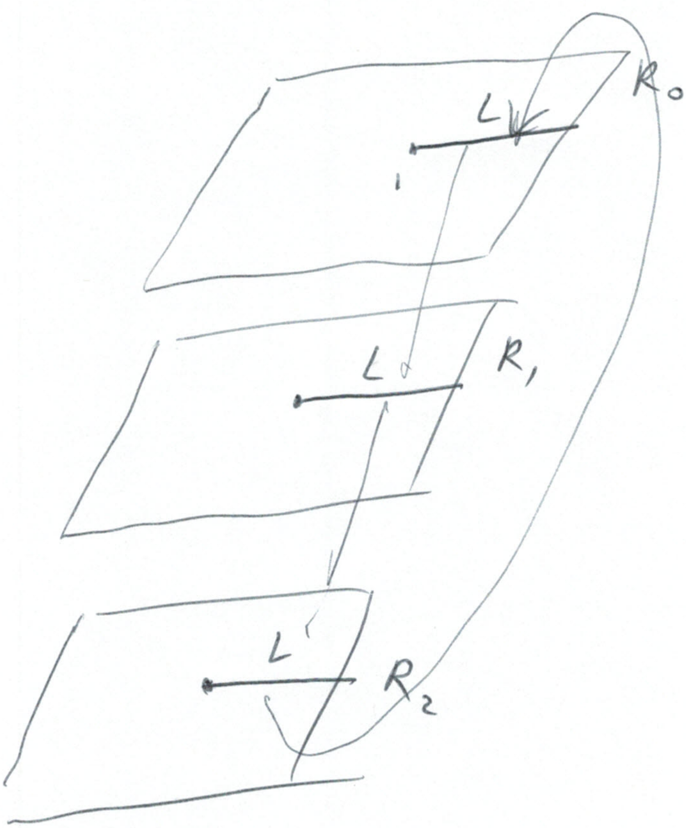
1

(1) $w = (z-1)^{1/3}$

Take branch cut at $z = [1, \infty)$

at upper edge of L
$w > 0$ for 1st branch,
add $e^{2\pi i/3}$ for 2nd branch
and $e^{4\pi i/3}$ for 3rd branch

3 sheets of Riemann surface: R_0, R_1, R_2



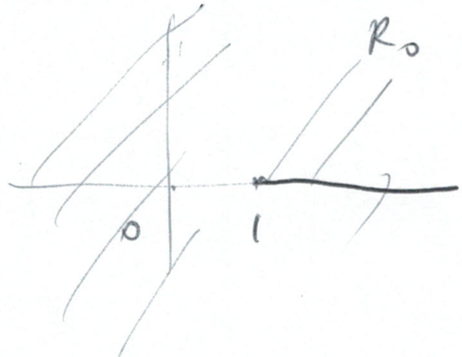
(1) Lower edge of L at R_0 is joined with upper edge of L at R_1

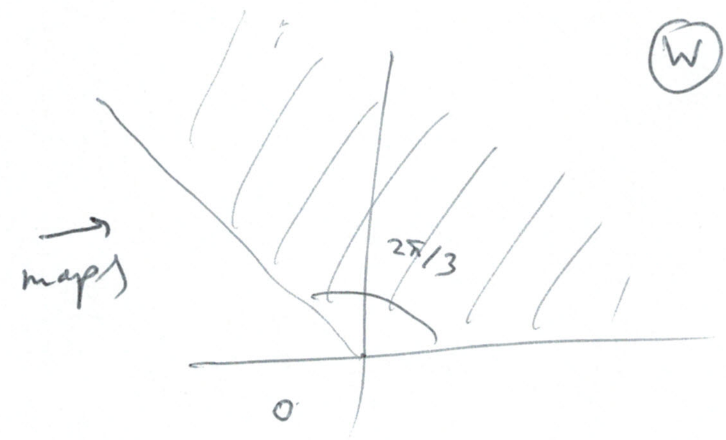
(2) Lower edge of L at R_1 is joined with upper edge of L at R_2

(3) Lower edge of L at R_2 is joined with upper edge of L at R_1

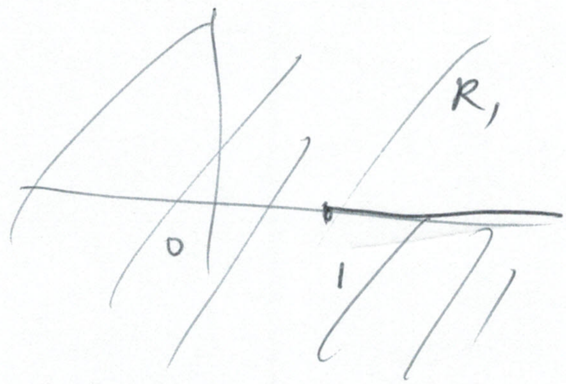
$$w = (z-1)^{1/3}$$

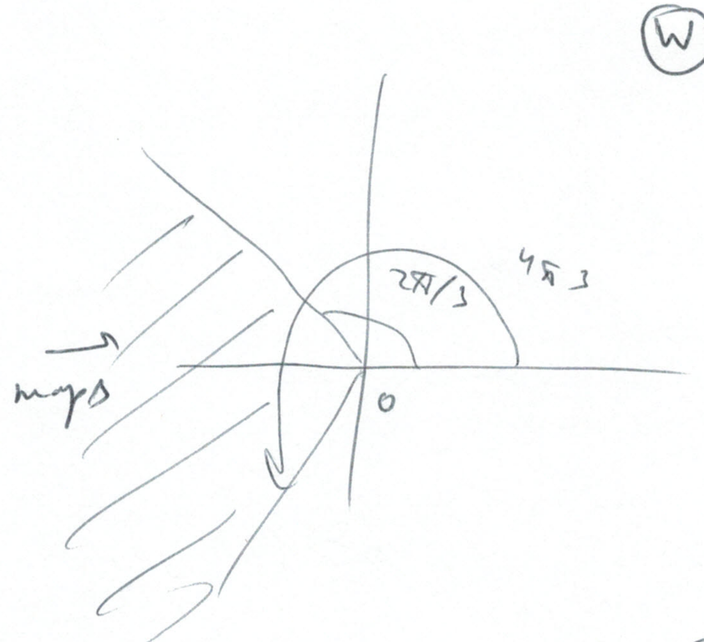
Sheet R_0



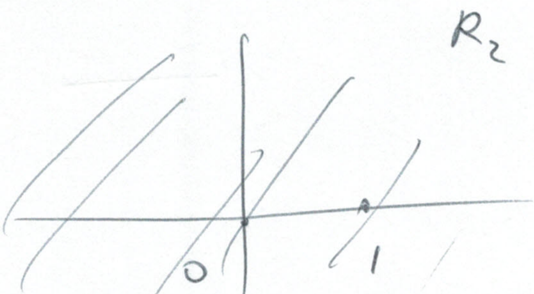


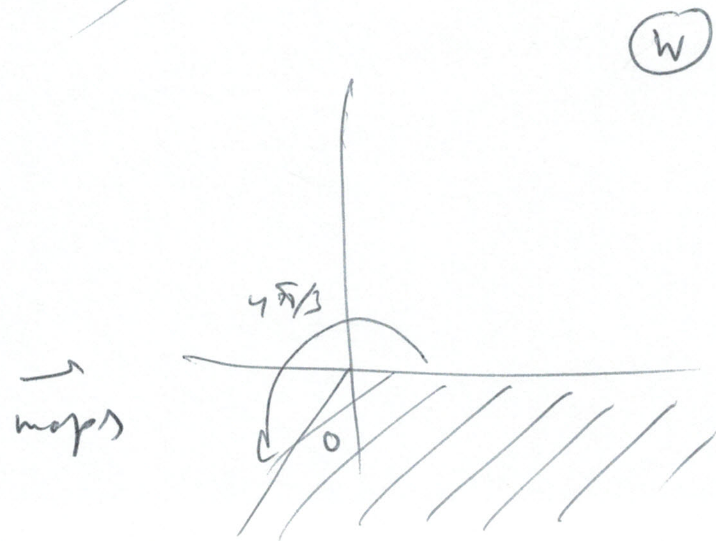
sheet R_1





sheet R_2





thus together all sheets

R_0, R_1, R_2 are mapped one-to-one

to the entire plane $\mathbb{C}$ of W.