EXERCISES

1. Show that

(a) $(1+i)^i = \exp\left(-\frac{\pi}{4} + 2n\pi\right)\exp\left(i\frac{\ln 2}{2}\right) \quad (n = 0, \pm 1, \pm 2, \ldots)$;

(b) $(-1)^{1/\pi} = e^{(2n+1)i} \quad (n = 0, \pm 1, \pm 2, \ldots)$.

2. Find the principal value of

(a) i^i; *(b)* $\left[\frac{e}{2}(-1-\sqrt{3}i)\right]^{3\pi i}$; *(c)* $(1-i)^{4i}$.

Ans. *(a)* $\exp(-\pi/2)$; *(b)* $-\exp(2\pi^2)$; *(c)* $e^{\pi}[\cos(2\ln 2) + i\sin(2\ln 2)]$.

3. Use definition (1), Sec. 33, of z^c to show that $(-1+\sqrt{3}i)^{3/2} = \pm 2\sqrt{2}$.

4. Show that the result in Exercise 3 could have been obtained by writing

(a) $(-1+\sqrt{3}i)^{3/2} = [(-1+\sqrt{3}i)^{1/2}]^3$ and first finding the square roots of $-1+\sqrt{3}i$;

(b) $(-1+\sqrt{3}i)^{3/2} = [(-1+\sqrt{3}i)^3]^{1/2}$ and first cubing $-1+\sqrt{3}i$.

5. Show that the *principal* nth root of a nonzero complex number z_0 that was defined in Sec. 9 is the same as the principal value of $z_0^{1/n}$ defined by equation (5), Sec. 33.

6. Show that if $z \neq 0$ and a is a real number, then $|z^a| = \exp(a\ln|z|) = |z|^a$, where the principal value of $|z|^a$ is to be taken.

7. Let $c = a + bi$ be a fixed complex number, where $c \neq 0, \pm 1, \pm 2, \ldots$, and note that i^c is multiple-valued. What additional restriction must be placed on the constant c so that the values of $|i^c|$ are all the same?

Ans. c is real.

8. Let c, c_1, c_2, and z denote complex numbers, where $z \neq 0$. Prove that if all of the powers involved are principal values, then

(a) $z^{c_1}z^{c_2} = z^{c_1+c_2}$; *(b)* $\frac{z^{c_1}}{z^{c_2}} = z^{c_1-c_2}$; *(c)* $(z^c)^n = z^{cn} \quad (n = 1, 2, \ldots)$.

9. Assuming that $f'(z)$ exists, state the formula for the derivative of $c^{f(z)}$.

34. TRIGONOMETRIC FUNCTIONS

Euler's formula (Sec. 6) tells us that

$$e^{ix} = \cos x + i\sin x \quad \text{and} \quad e^{-ix} = \cos x - i\sin x$$

for every real number x. Hence

$$e^{ix} - e^{-ix} = 2i\sin x \quad \text{and} \quad e^{ix} + e^{-ix} = 2\cos x.$$

That is,

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \quad \text{and} \quad \cos x = \frac{e^{ix} + e^{-ix}}{2}.$$