EXERCISES

1. Give details in the derivation of expressions (2), Sec. 34, for the derivatives of $\sin z$ and $\cos z$.

2. *(a)* With the aid of expression (4), Sec. 34, show that

$$e^{iz_1}e^{iz_2} = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 + i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2).$$

Then use relations (3), Sec. 34, to show how it follows that

$$e^{-iz_1}e^{-iz_2} = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 - i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2).$$

(b) Use the results in part *(a)* and the fact that

$$\sin(z_1 + z_2) = \frac{1}{2i}\left[e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}\right] = \frac{1}{2i}\left(e^{iz_1}e^{iz_2} - e^{-iz_1}e^{-iz_2}\right)$$

to obtain the identity

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$$

in Sec. 34.

3. According to the final result in Exercise 2*(b)*,

$$\sin(z + z_2) = \sin z \cos z_2 + \cos z \sin z_2.$$

By differentiating each side here with respect to z and then setting $z = z_1$, derive the expression

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$$

that was stated in Sec. 34.

4. Verify identity (9) in Sec. 34 using

(a) identity (6) and relations (3) in that section;

(b) the lemma in Sec. 27 and the fact that the entire function

$$f(z) = \sin^2 z + \cos^2 z - 1$$

has zero values along the x axis.

5. Use identity (9) in Sec. 34 to show that

(a) $1 + \tan^2 z = \sec^2 z$; *(b)* $1 + \cot^2 z = \csc^2 z$.

6. Establish differentiation formulas (21) and (22) in Sec. 34.

7. In Sec. 34, use expressions (13) and (14) to derive expressions (15) and (16) for $|\sin z|^2$ and $|\cos z|^2$.

Suggestion: Recall the identities $\sin^2 x + \cos^2 x = 1$ and $\cosh^2 y - \sinh^2 y = 1$.

8. Point out how it follows from expressions (15) and (16) in Sec. 34 for $|\sin z|^2$ and $|\cos z|^2$ that

(a) $|\sin z| \geq |\sin x|$; *(b)* $|\cos z| \geq |\cos x|$.

9. With the aid of expressions (15) and (16) in Sec. 34 for $|\sin z|^2$ and $|\cos z|^2$, show that

(a) $|\sinh y| \le |\sin z| \le \cosh y$; *(b)* $|\sinh y| \le |\cos z| \le \cosh y$.

10. *(a)* Use definitions (1), Sec. 34, of $\sin z$ and $\cos z$ to show that

$$2\sin(z_1 + z_2)\sin(z_1 - z_2) = \cos 2z_2 - \cos 2z_1.$$

(b) With the aid of the identity obtained in part *(a)*, show that if $\cos z_1 = \cos z_2$, then at least one of the numbers $z_1 + z_2$ and $z_1 - z_2$ is an integral multiple of 2π.

11. Use the Cauchy–Riemann equations and the theorem in Sec. 21 to show that neither $\sin \overline{z}$ nor $\cos \overline{z}$ is an analytic function of z anywhere.

12. Use the reflection principle (Sec. 28) to show that for all z,

(a) $\overline{\sin z} = \sin \overline{z}$; *(b)* $\overline{\cos z} = \cos \overline{z}$.

13. With the aid of expressions (13) and (14) in Sec. 34, give direct verifications of the relations obtained in Exercise 12.

14. Show that

(a) $\overline{\cos(iz)} = \cos(i\overline{z})$ for all z;

(b) $\overline{\sin(iz)} = \sin(i\overline{z})$ if and only if $z = n\pi i \; (n = 0, \pm 1, \pm 2, \ldots)$.

15. Find all roots of the equation $\sin z = \cosh 4$ by equating the real parts and then the imaginary parts of $\sin z$ and $\cosh 4$.

Ans. $\left(\dfrac{\pi}{2} + 2n\pi\right) \pm 4i \qquad (n = 0, \pm 1, \pm 2, \ldots)$.

16. With the aid of expression (14), Sec. 34, show that the roots of the equaion $\cos z = 2$ are

$$z = 2n\pi + i\cosh^{-1} 2 \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Then express them in the form

$$z = 2n\pi \pm i\ln(2 + \sqrt{3}) \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

35. HYPERBOLIC FUNCTIONS

The hyperbolic sine and the hyperbolic cosine of a complex variable are defined as they are with a real variable; that is,

$$\sinh z = \frac{e^z - e^{-z}}{2}, \quad \cosh z = \frac{e^z + e^{-z}}{2}. \tag{1}$$

Since e^z and e^{-z} are entire, it follows from definitions (1) that $\sinh z$ and $\cosh z$ are entire. Furthermore,

$$\frac{d}{dz}\sinh z = \cosh z, \quad \frac{d}{dz}\cosh z = \sinh z. \tag{2}$$