In view of the periodicity of $\sin z$ and $\cos z$, it follows immediately from relaions (4) that $\sinh z$ and $\cosh z$ are periodic with period $2\pi i$. Relations (4), together with statements (17) and (18) in Sec. 34, also tell us that

$$\sinh z = 0 \quad \text{if and only if} \quad z = n\pi i \;\; (n = 0, \pm 1, \pm 2, \ldots) \tag{14}$$

and

$$\cosh z = 0 \quad \text{if and only if} \quad z = \left(\frac{\pi}{2} + n\pi\right) i \;\; (n = 0, \pm 1, \pm 2, \ldots). \tag{15}$$

The hyperbolic tangent of z is defined by means of the equation

$$\tanh z = \frac{\sinh z}{\cosh z} \tag{16}$$

and is analytic in every domain in which $\cosh z \neq 0$. The functions $\coth z$, $\operatorname{sech} z$, and $\operatorname{csch} z$ are the reciprocals of $\tanh z$, $\cosh z$, and $\sinh z$, respectively. It is straightforward to verify the following differentiation formulas, which are the same as those established in calculus for the corresponding functions of a real variable:

$$\frac{d}{dz} \tanh z = \operatorname{sech}^2 z, \qquad \frac{d}{dz} \coth z = -\operatorname{csch}^2 z, \tag{17}$$

$$\frac{d}{dz} \operatorname{sech} z = -\operatorname{sech} z \tanh z, \qquad \frac{d}{dz} \operatorname{csch} z = -\operatorname{csch} z \coth z. \tag{18}$$

EXERCISES

1. Verify that the derivatives of $\sinh z$ and $\cosh z$ are as stated in equations (2), Sec. 35.

2. Prove that $\sinh 2z = 2 \sinh z \cosh z$ by starting with

(a) definitions (1), Sec. 35, of $\sinh z$ and $\cosh z$;

(b) the identity $\sin 2z = 2 \sin z \cos z$ (Sec. 34) and using relations (3) in Sec. 35.

3. Show how identities (6) and (8) in Sec. 35 follow from identities (9) and (6), respectively, in Sec. 34.

4. Write $\sinh z = \sinh(x + iy)$ and $\cosh z = \cosh(x + iy)$, and then show how expressions (9) and (10) in Sec. 35 follow from identities (7) and (8), respectively, in that section.

5. Verify expression (12), Sec. 35, for $|\cosh z|^2$.

6. Show that $|\sinh x| \leq |\cosh z| \leq \cosh x$ by using

(a) identity (12), Sec. 35;

(b) the inequalities $|\sinh y| \leq |\cos z| \leq \cosh y$, obtained in Exercise 9*(b)*, Sec. 34.

7. Show that

(a) $\sinh(z + \pi i) = -\sinh z$; *(b)* $\cosh(z + \pi i) = -\cosh z$;
(c) $\tanh(z + \pi i) = \tanh z$.

8. Give details showing that the zeros of $\sinh z$ and $\cosh z$ are as in statements (14) and (l5), Sec. 35.

9. Using the results proved in Exercise 8, locate all zeros and singularities of the hyperbolic tangent function.

10. Derive differentiation formulas (17), Sec. 35.

11. Use the reflection principle (Sec. 28) to show that for all z,

(a) $\overline{\sinh z} = \sinh \overline{z}$; *(b)* $\overline{\cosh z} = \cosh \overline{z}$.

12. Use the results in Exercise 11 to show that $\overline{\tanh z} = \tanh \overline{z}$ at points where $\cosh z \neq 0$.

13. By accepting that the stated identity is valid when z is replaced by the real variable x and using the lemma in Sec. 27, verify that

(a) $\cosh^2 z - \sinh^2 z = 1$; *(b)* $\sinh z + \cosh z = e^z$.

[Compare with Exercise 4*(b)*, Sec. 34.]

14. Why is the function $\sinh(e^z)$ entire? Write its real component as a function of x and y, and state why that function must be harmonic everywhere.

15. By using one of the identities (9) and (10) in Sec. 35 and then proceeding as in Exercise 15, Sec. 34, find all roots of the equation

(a) $\sinh z = i$; *(b)* $\cosh z = \dfrac{1}{2}$.

Ans. (a) $z = \left(2n + \dfrac{1}{2}\right)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots)$;

(b) $z = \left(2n \pm \dfrac{1}{3}\right)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots)$.

16. Find all roots of the equation $\cosh z = -2$. (Compare this exercise with Exercise 16, Sec. 34.)

Ans. $z = \pm \ln(2 + \sqrt{3}) + (2n + 1)\pi i \ (n = 0, \pm 1, \pm 2, \ldots)$.

36. INVERSE TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

Inverses of the trigonometric and hyperbolic functions can be described in terms of logarithms.

In order to define the inverse sine function $\sin^{-1} z$, we write

$$w = \sin^{-1} z \quad \text{when} \quad z = \sin w.$$

That is, $w = \sin^{-1} z$ when

$$z = \frac{e^{iw} - e^{-iw}}{2i}.$$

If we put this equation in the form

$$(e^{iw})^2 - 2iz(e^{iw}) - 1 = 0,$$