become single-valued and analytic because they are then compositions of analytic functions.

The derivatives of these three functions are readily obtained from their logarithmic expressions. The derivatives of the first two depend on the values chosen for the square roots:

$$\frac{d}{dz}\sin^{-1} z = \frac{1}{(1-z^2)^{1/2}}, \tag{5}$$

$$\frac{d}{dz}\cos^{-1} z = \frac{-1}{(1-z^2)^{1/2}}. \tag{6}$$

The derivative of the last one,

$$\frac{d}{dz}\tan^{-1} z = \frac{1}{1+z^2}, \tag{7}$$

does not, however, depend on the manner in which the function is made single-valued.

Inverse hyperbolic functions can be treated in a corresponding manner. It turns out that

$$\sinh^{-1} z = \log\left[z + (z^2+1)^{1/2}\right], \tag{8}$$

$$\cosh^{-1} z = \log\left[z + (z^2-1)^{1/2}\right], \tag{9}$$

and

$$\tanh^{-1} z = \frac{1}{2}\log\frac{1+z}{1-z}. \tag{10}$$

Finally, we remark that common alternative notation for all of these inverse functions is arcsin z, etc.

EXERCISES

1. Find all the values of

(a) $\tan^{-1}(2i)$; *(b)* $\tan^{-1}(1+i)$; *(c)* $\cosh^{-1}(-1)$; *(d)* $\tanh^{-1} 0$.

Ans. *(a)* $\left(n+\frac{1}{2}\right)\pi + \frac{i}{2}\ln 3 \;(n = 0, \pm 1, \pm 2, \ldots)$;

(d) $n\pi i \;(n = 0, \pm 1, \pm 2, \ldots)$.

2. Solve the equation $\sin z = 2$ for z by

(a) equating real parts and then imaginary parts in that equation;

(b) using expression (2), Sec. 36, for $\sin^{-1} z$.

Ans. $z = \left(2n+\frac{1}{2}\right)\pi \pm i\ln(2+\sqrt{3}) \quad (n = 0, \pm 1, \pm 2, \ldots)$.

3. Solve the equation $\cos z = \sqrt{2}$ for z.

4. Derive formula (5), Sec. 36, for the derivative of $\sin^{-1} z$.

5. Derive expression (4), Sec. 36, for $\tan^{-1} z$.

6. Derive formula (7), Sec. 36, for the derivative of $\tan^{-1} z$.

7. Derive expression (9), Sec. 36, for $\cosh^{-1} z$.