EXAMPLE 3. If a point z lies on the unit circle $|z| = 1$ about the origin, it follows from inequalities (7) and (8) that

$$|z - 2| \le |z| + 2 = 3$$

and

$$|z - 2| \ge ||z| - 2| = 1.$$

The triangle inequality (4) can be generalized by means of mathematical induction to sums involving any finite number of terms:

$$|z_1 + z_2 + \cdots + z_n| \le |z_1| + |z_2| + \cdots + |z_n| \qquad (n = 2, 3, \ldots). \tag{10}$$

To give details of the induction proof here, we note that when $n = 2$, inequality (10) is just inequality (4). Furthermore, if inequality (10) is assumed to be valid when $n = m$, it must also hold when $n = m + 1$ since, by inequality (4),

$$\begin{aligned}|(z_1 + z_2 + \cdots + z_m) + z_{m+1}| &\le |z_1 + z_2 + \cdots + z_m| + |z_{m+1}| \\ &\le (|z_1| + |z_2| + \cdots + |z_m|) + |z_{m+1}|.\end{aligned}$$

EXERCISES

1. Locate the numbers $z_1 + z_2$ and $z_1 - z_2$ vectorially when

(a) $z_1 = 2i, \quad z_2 = \dfrac{2}{3} - i$; *(b)* $z_1 = (-\sqrt{3}, 1), \quad z_2 = (\sqrt{3}, 0)$;
(c) $z_1 = (-3, 1), \quad z_2 = (1, 4)$; *(d)* $z_1 = x_1 + iy_1, \quad z_2 = x_1 - iy_1$.

2. Verify inequalities (3), Sec. 4, involving $\operatorname{Re} z$, $\operatorname{Im} z$, and $|z|$.

3. Use established properties of moduli to show that when $|z_3| \ne |z_4|$,

$$\frac{\operatorname{Re}(z_1 + z_2)}{|z_3 + z_4|} \le \frac{|z_1| + |z_2|}{||z_3| - |z_4||}.$$

4. Verify that $\sqrt{2}\,|z| \ge |\operatorname{Re} z| + |\operatorname{Im} z|$.

Suggestion: Reduce this inequality to $(|x| - |y|)^2 \ge 0$.

5. In each case, sketch the set of points determined by the given condition:

(a) $|z - 1 + i| = 1$; *(b)* $|z + i| \le 3$; *(c)* $|z - 4i| \ge 4$.

6. Using the fact that $|z_1 - z_2|$ is the distance between two points z_1 and z_2, give a geometric argument that

(a) $|z - 4i| + |z + 4i| = 10$ represents an ellipse whose foci are $(0, \pm 4)$;
(b) $|z - 1| = |z + i|$ represents the line through the origin whose slope is -1.