EXERCISES

1. Use rules in calculus to establish the following rules when

$$w(t) = u(t) + iv(t)$$

is a complex-valued function of a real variable t and $w'(t)$ exists:

(a) $\dfrac{d}{dt} w(-t) = -w'(-t)$ where $w'(-t)$ denotes the derivative of $w(t)$ with respect to t, evaluated at $-t$;

(b) $\dfrac{d}{dt}[w(t)]^2 = 2\,w(t)w'(t)$.

2. Evaluate the following integrals:

(a) $\displaystyle\int_1^2 \left(\frac{1}{t} - i\right)^2 dt$; *(b)* $\displaystyle\int_0^{\pi/6} e^{i2t}\, dt$; *(c)* $\displaystyle\int_0^{\infty} e^{-zt}\, dt$ $\quad(\operatorname{Re} z > 0)$.

Ans. *(a)* $-\dfrac{1}{2} - i\ln 4$; *(b)* $\dfrac{\sqrt{3}}{4} + \dfrac{i}{4}$; *(c)* $\dfrac{1}{z}$.

3. Show that if m and n are integers,

$$\int_0^{2\pi} e^{im\theta} e^{-in\theta}\, d\theta = \begin{cases} 0 & \text{when } m \neq n, \\ 2\pi & \text{when } m = n. \end{cases}$$

4. According to definition (2), Sec. 38, of definite integrals of complex-valued functions of a real variable,

$$\int_0^{\pi} e^{(1+i)x}\, dx = \int_0^{\pi} e^x \cos x\, dx + i \int_0^{\pi} e^x \sin x\, dx.$$

Evaluate the two integrals on the right here by evaluating the single integral on the left and then using the real and imaginary parts of the value found.

Ans. $-(1 + e^{\pi})/2, \quad (1 + e^{\pi})/2$.

5. Let $w(t) = u(t) + iv(t)$ denote a continuous complex-valued function defined on an interval $-a \le t \le a$.

(a) Suppose that $w(t)$ is *even*; that is, $w(-t) = w(t)$ for each point t in the given interval. Show that

$$\int_{-a}^{a} w(t)\, dt = 2\int_0^{a} w(t)\, dt.$$

(b) Show that if $w(t)$ is an *odd* function, one where $w(-t) = -w(t)$ for each point t in the given interval, then

$$\int_{-a}^{a} w(t)\, dt = 0.$$

Suggestion: In each part of this exercise, use the corresponding property of integrals of *real-valued* functions of t, which is graphically evident.