$a < t < b$. This expression for **T** is the one learned in calculus when $z(t)$ is interpreted as a radius vector. Such an arc is said to be *smooth*. In referring to a smooth arc $z = z(t)$ $(a \le t \le b)$, then, we agree that the derivative $z'(t)$ is continuous on the closed interval $a \le t \le b$ and nonzero throughout the open interval $a < t < b$.

A *contour*, or piecewise smooth arc, is an arc consisting of a finite number of smooth arcs joined end to end. Hence if equation (2) represents a contour, $z(t)$ is continuous, whereas its derivative $z'(t)$ is piecewise continuous. The polygonal line (4) is, for example, a contour. When only the initial and final values of $z(t)$ are the same, a contour C is called a *simple closed contour*. Examples are the circles (5) and (6), as well as the boundary of a triangle or a rectangle taken in a specific direction. The length of a contour or a simple closed contour is the sum of the lengths of the smooth arcs that make up the contour.

The points on any simple closed curve or simple closed contour C are boundary points of two distinct domains, one of which is the interior of C and is bounded. The other, which is the exterior of C, is unbounded. It will be convenient to accept this statement, known as the *Jordan curve theorem*, as geometrically evident; the proof is not easy.*

EXERCISES

1. Show that if $w(t) = u(t) + iv(t)$ is continuous on an interval $a \le t \le b$, then

(a) $\displaystyle\int_{-b}^{-a} w(-t)\,dt = \int_a^b w(\tau)\,d\tau$;

(b) $\displaystyle\int_a^b w(t)\,dt = \int_\alpha^\beta w[\phi(\tau)]\phi'(\tau)\,d\tau$, where $\phi(\tau)$ is the function in equation (9), Sec. 39.

Suggestion: These identities can be obtained by noting that they are valid for *real-valued* functions of t.

2. Let C denote the right-hand half of the circle $|z| = 2$, in the counterclockwise direction, and note that two parametric representations for C are

$$z = z(\theta) = 2\,e^{i\theta} \qquad \left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right)$$

and

$$z = Z(y) = \sqrt{4 - y^2} + iy \qquad (-2 \le y \le 2).$$

Verify that $Z(y) = z[\phi(y)]$, where

$$\phi(y) = \arctan \frac{y}{\sqrt{4 - y^2}} \qquad \left(-\frac{\pi}{2} < \arctan t < \frac{\pi}{2}\right).$$

*See pp. 115–116 of the book by Newman or Sec. 13 of the one by Thron, both of which are cited in Appendix 1. The special case in which C is a simple closed polygon is proved on pp. 281–285 of Vol. 1 of the work by Hille, also cited in Appendix 1.

Also, show that this function ϕ has a positive derivative, as required in the conditions following equation (9), Sec. 39.

3. Derive the equation of the line through the points (α, a) and (β, b) in the τt plane that are shown in Fig. 37. Then use it to find the linear function $\phi(\tau)$ which can be used in equation (9), Sec. 39, to transform representation (2) in that section into representation (10) there.

Ans. $\phi(\tau) = \dfrac{b-a}{\beta-\alpha}\tau + \dfrac{a\beta - b\alpha}{\beta - \alpha}.$

4. Verify expression (14), Sec. 39, for the derivative of $Z(\tau) = z[\phi(\tau)]$.

Suggestion: Write $Z(\tau) = x[\phi(\tau)] + iy[\phi(\tau)]$ and apply the chain rule for real-valued functions of a real variable.

5. Suppose that a function $f(z)$ is analytic at a point $z_0 = z(t_0)$ lying on a smooth arc $z = z(t)$ $(a \le t \le b)$. Show that if $w(t) = f[z(t)]$, then

$$w'(t) = f'[z(t)]z'(t)$$

when $t = t_0$.

Suggestion: Write $f(z) = u(x, y) + iv(x, y)$ and $z(t) = x(t) + iy(t)$, so that

$$w(t) = u[x(t), y(t)] + iv[x(t), y(t)].$$

Then apply the chain rule in calculus for functions of two real variables to write

$$w' = (u_x x' + u_y y') + i(v_x x' + v_y y'),$$

and use the Cauchy–Riemann equations.

6. Let $y(x)$ be a real-valued function defined on the interval $0 \le x \le 1$ by means of the equations

$$y(x) = \begin{cases} x^3 \sin(\pi/x) & \text{when } 0 < x \le 1, \\ 0 & \text{when } x = 0. \end{cases}$$

(a) Show that the equation

$$z = x + iy(x) \qquad (0 \le x \le 1)$$

represents an arc C that intersects the real axis at the points $z = 1/n$ $(n = 1, 2, \ldots)$ and $z = 0$, as shown in Fig. 38.

(b) Verify that the arc C in part *(a)* is, in fact, a *smooth* arc.

Suggestion: To establish the continuity of $y(x)$ at $x = 0$, observe that

$$0 \le \left| x^3 \sin\left(\frac{\pi}{x}\right)\right| \le x^3$$

when $x > 0$. A similar remark applies in finding $y'(0)$ and showing that $y'(x)$ is continuous at $x = 0$.

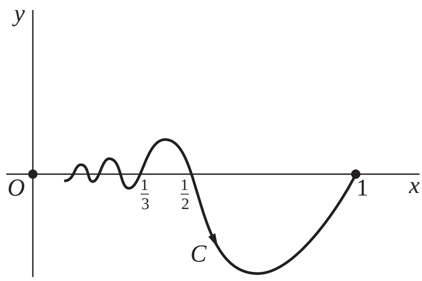

FIGURE 38

40. CONTOUR INTEGRALS

We turn now to integrals of complex-valued functions f of the complex variable z. Such an integral is defined in terms of the values $f(z)$ along a given contour C, extending from a point $z = z_1$ to a point $z = z_2$ in the complex plane. It is, therefore, a line integral; and its value depends, in general, on the contour C as well as on the function f. It is written

$$\int_C f(z)\,dz \quad \text{or} \quad \int_{z_1}^{z_2} f(z)\,dz,$$

the latter notation often being used when the value of the integral is independent of the choice of the contour taken between two fixed end points. While the integral may be defined directly as the limit of a sum, we choose to define it in terms of a definite integral of the type introduced in Sec. 38.

Suppose that the equation

(1) $$z = z(t) \quad (a \le t \le b)$$

represents a contour C, extending from a point $z_1 = z(a)$ to a point $z_2 = z(b)$. We assume that $f[z(t)]$ is *piecewise continuous* (Sec. 38) on the interval $a \le t \le b$ and refer to the function $f(z)$ as being piecewise continuous on C. We then define the line integral, or *contour integral*, of f along C in terms of the parameter t:

(2) $$\int_C f(z)\,dz = \int_a^b f[z(t)]z'(t)\,dt.$$

Note that since C is a contour, $z'(t)$ is also piecewise continuous on $a \le t \le b$; and so the existence of integral (2) is ensured.

The value of a contour integral is invariant under a change in the representation of its contour when the change is of the type (11), Sec. 39. This can be seen by following the same general procedure that was used in Sec. 39 to show the invariance of arc length.