Note that if a is a nonzero integer n, this result tells us that

(5) $$\int_C z^{n-1}\,dz = 0 \quad (n = \pm 1, \pm 2, \ldots).$$

If a is allowed to be zero, we have

(6) $$\int_C \frac{dz}{z} = \int_{-\pi}^{\pi} \frac{1}{Re^{i\theta}} iRe^{i\theta}\,d\theta = i\int_{-\pi}^{\pi} d\theta = 2\pi i.$$

EXERCISES

For the functions f and contours C in Exercises 1 through 7, use parametric representations for C, or legs of C, to evaluate

$$\int_C f(z)\,dz.$$

1. $f(z) = (z+2)/z$ and C is

(a) the semicircle $z = 2e^{i\theta}$ $(0 \le \theta \le \pi)$;
(b) the semicircle $z = 2e^{i\theta}$ $(\pi \le \theta \le 2\pi)$;
(c) the circle $z = 2e^{i\theta}$ $(0 \le \theta \le 2\pi)$.

Ans. (a) $-4 + 2\pi i$; *(b)* $4 + 2\pi i$; *(c)* $4\pi i$.

2. $f(z) = z - 1$ and C is the arc from $z = 0$ to $z = 2$ consisting of

(a) the semicircle $z = 1 + e^{i\theta}$ $(\pi \le \theta \le 2\pi)$;
(b) the segment $z = x$ $(0 \le x \le 2)$ of the real axis.

Ans. (a) 0; *(b)* 0.

3. $f(z) = \pi \exp(\pi \bar{z})$ and C is the boundary of the square with vertices at the points $0, 1, 1+i$, and i, the orientation of C being in the counterclockwise direction.

Ans. $4(e^{\pi} - 1)$.

4. $f(z)$ is defined by means of the equations

$$f(z) = \begin{cases} 1 & \text{when } y < 0, \\ 4y & \text{when } y > 0, \end{cases}$$

and C is the arc from $z = -1 - i$ to $z = 1 + i$ along the curve $y = x^3$.

Ans. $2 + 3i$.

5. $f(z) = 1$ and C is an arbitrary contour from any fixed point z_1 to any fixed point z_2 in the z plane.

Ans. $z_2 - z_1$.

6. $f(z)$ is the branch

$$z^{-1+i} = \exp[(-1+i)\log z] \qquad (|z| > 0, 0 < \arg z < 2\pi)$$

of the indicated power function, and C is the unit circle $z = e^{i\theta}$ $(0 \le \theta \le 2\pi)$.

Ans. $i(1 - e^{-2\pi})$.

7. $f(z)$ is the principal branch

$$z^i = \exp(i\,\text{Log}\, z) \quad (|z| > 0, -\pi < \text{Arg}\, z < \pi)$$

of this power function, and C is the semicircle $z = e^{i\theta}$ $(0 \le \theta \le \pi)$.

Ans. $-\dfrac{1+e^{-\pi}}{2}(1-i)$.

8. With the aid of the result in Exercise 3, Sec. 38, evaluate the integral

$$\int_C z^m \,\overline{z}^{\,n}\, dz,$$

where m and n are integers and C is the unit circle $|z| = 1$, taken counterclockwise.

9. Evaluate the integral I in Example 1, Sec. 41, using this representation for C:

$$z = \sqrt{4 - y^2} + iy \qquad (-2 \le y \le 2).$$

(See Exercise 2, Sec. 39.)

10. Let C_0 and C denote the circles

$$z = z_0 + Re^{i\theta}\ (-\pi \le \theta \le \pi) \quad \text{and} \quad z = Re^{i\theta}\ (-\pi \le \theta \le \pi),$$

respectively.

(a) Use these parametric representations to show that

$$\int_{C_0} f(z - z_0)\, dz = \int_C f(z)\, dz$$

when f is piecewise continuous on C.

(b) Apply the result in part *(a)* to integrals (5) and (6) in Sec. 42 to show that

$$\int_{C_0} (z - z_0)^{n-1}\, dz = 0 \ (n = \pm 1, \pm 2, \ldots) \quad \text{and} \quad \int_{C_0} \frac{dz}{z - z_0} = 2\pi i.$$

11. *(a)* Suppose that a function $f(z)$ is continuous on a smooth arc C, which has a parametric representation $z = z(t)$ $(a \le t \le b)$; that is, $f[z(t)]$ is continuous on the interval $a \le t \le b$. Show that if $\phi(\tau)$ $(\alpha \le \tau \le \beta)$ is the function described in Sec. 39, then

$$\int_a^b f[z(t)]z'(t)\, dt = \int_\alpha^\beta f[Z(\tau)]Z'(\tau)\, d\tau$$

where $Z(\tau) = z[\phi(\tau)]$.

(b) Point out how it follows that the identity obtained in part *(a)* remains valid when C is any contour, not necessarily a smooth one, and $f(z)$ is piecewise continuous on C. Thus show that the value of the integral of $f(z)$ along C is the same when the representation $z = Z(\tau)$ $(\alpha \le \tau \le \beta)$ is used, instead of the original one.

Suggestion: In part *(a)*, use the result in Exercise 1*(b)*, Sec. 39, and then refer to expression (14) in that section.