An important identity relating the conjugate of a complex number $z = x + iy$ to its modulus is

$$z\,\overline{z} = |z|^2, \tag{7}$$

where each side is equal to $x^2 + y^2$. It suggests the method for determining a quotient z_1/z_2 that begins with expression (7), Sec. 3. That method is, of course, based on multiplying both the numerator and the denominator of z_1/z_2 by $\overline{z_2}$, so that the denominator becomes the real number $|z_2|^2$.

EXAMPLE 1. As an illustration,

$$\frac{-1+3i}{2-i} = \frac{(-1+3i)(2+i)}{(2-i)(2+i)} = \frac{-5+5i}{|2-i|^2} = \frac{-5+5i}{5} = -1+i.$$

See also the example in Sec. 3.

Identity (7) is especially useful in obtaining properties of moduli from properties of conjugates noted above. We mention that

$$|z_1 z_2| = |z_1||z_2| \tag{8}$$

and

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \qquad (z_2 \neq 0). \tag{9}$$

Property (8) can be established by writing

$$|z_1 z_2|^2 = (z_1 z_2)(\overline{z_1 z_2}) = (z_1 z_2)(\overline{z_1}\,\overline{z_2}) = (z_1\overline{z_1})(z_2\overline{z_2}) = |z_1|^2|z_2|^2 = (|z_1||z_2|)^2$$

and recalling that a modulus is never negative. Property (9) can be verified in a similar way.

EXAMPLE 2. Property (8) tells us that $|z^2| = |z|^2$ and $|z^3| = |z|^3$. Hence if z is a point inside the circle centered at the origin with radius 2, so that $|z| < 2$, it follows from the generalized triangle inequality (10) in Sec. 4 that

$$|z^3 + 3z^2 - 2z + 1| \leq |z|^3 + 3|z|^2 + 2|z| + 1 < 25.$$

EXERCISES

1. Use properties of conjugates and moduli established in Sec. 5 to show that

(a) $\overline{\overline{z} + 3i} = z - 3i$; *(b)* $\overline{iz} = -i\overline{z}$;

(c) $\overline{(2+i)^2} = 3 - 4i$; *(d)* $|(2\overline{z}+5)(\sqrt{2} - i)| = \sqrt{3}\,|2z+5|$.

2. Sketch the set of points determined by the condition

(a) $\text{Re}(\overline{z} - i) = 2$; *(b)* $|2\overline{z} + i| = 4$.

3. Verify properties (3) and (4) of conjugates in Sec. 5.

4. Use property (4) of conjugates in Sec. 5 to show that

(a) $\overline{z_1 z_2 z_3} = \overline{z_1}\,\overline{z_2}\,\overline{z_3}$; (b) $\overline{z^4} = \overline{z}^4$.

5. Verify property (9) of moduli in Sec. 5.

6. Use results in Sec. 5 to show that when z_2 and z_3 are nonzero,

(a) $\overline{\left(\dfrac{z_1}{z_2 z_3}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}\,\overline{z_3}}$; (b) $\left|\dfrac{z_1}{z_2 z_3}\right| = \dfrac{|z_1|}{|z_2||z_3|}$.

7. Show that

$$|\text{Re}(2 + \overline{z} + z^3)| \le 4 \qquad \text{when } |z| \le 1.$$

8. It is shown in Sec. 3 that if $z_1 z_2 = 0$, then at least one of the numbers z_1 and z_2 must be zero. Give an alternative proof based on the corresponding result for real numbers and using identity (8), Sec. 5.

9. By factoring $z^4 - 4z^2 + 3$ into two quadratic factors and using inequality (8), Sec. 4, show that if z lies on the circle $|z| = 2$, then

$$\left|\frac{1}{z^4 - 4z^2 + 3}\right| \le \frac{1}{3}.$$

10. Prove that

(a) z is real if and only if $\overline{z} = z$;

(b) z is either real or pure imaginary if and only if $\overline{z}^2 = z^2$.

11. Use mathematical induction to show that when $n = 2, 3, \ldots,$

(a) $\overline{z_1 + z_2 + \cdots + z_n} = \overline{z_1} + \overline{z_2} + \cdots + \overline{z_n}$; (b) $\overline{z_1 z_2 \cdots z_n} = \overline{z_1}\,\overline{z_2} \cdots \overline{z_n}$.

12. Let $a_0, a_1, a_2, \ldots, a_n$ $(n \ge 1)$ denote *real* numbers, and let z be any complex number. With the aid of the results in Exercise 11, show that

$$\overline{a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n} = a_0 + a_1 \overline{z} + a_2 \overline{z}^2 + \cdots + a_n \overline{z}^{\,n}.$$

13. Show that the equation $|z - z_0| = R$ of a circle, centered at z_0 with radius R, can be written

$$|z|^2 - 2\,\text{Re}(z\overline{z_0}) + |z_0|^2 = R^2.$$

14. Using expressions (6), Sec. 5, for $\text{Re}\, z$ and $\text{Im}\, z$, show that the hyperbola $x^2 - y^2 = 1$ can be written

$$z^2 + \overline{z}^2 = 2.$$

15. Follow the steps below to give an algebraic derivation of the triangle inequality (Sec. 4)

$$|z_1 + z_2| \le |z_1| + |z_2|.$$

(a) Show that

$$|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1} + \overline{z_2}) = z_1\overline{z_1} + (z_1\overline{z_2} + \overline{z_1\overline{z_2}}) + z_2\overline{z_2}.$$

(b) Point out why

$$z_1\overline{z_2} + \overline{z_1\overline{z_2}} = 2\,\mathrm{Re}(z_1\overline{z_2}) \le 2|z_1||z_2|.$$

(c) Use the results in parts *(a)* and *(b)* to obtain the inequality

$$|z_1 + z_2|^2 \le (|z_1| + |z_2|)^2,$$

and note how the triangle inequality follows.

6. EXPONENTIAL FORM

Let r and θ be polar coordinates of the point (x, y) that corresponds to a *nonzero* complex number $z = x + iy$. Since $x = r\cos\theta$ and $y = r\sin\theta$, the number z can be written in *polar form* as

$$z = r(\cos\theta + i\sin\theta). \tag{1}$$

If $z = 0$, the coordinate θ is undefined; and so it is understood that $z \ne 0$ whenever polar coordinates are used.

In complex analysis, the real number r is not allowed to be negative and is the length of the radius vector for z; that is, $r = |z|$. The real number θ represents the angle, measured in radians, that z makes with the positive real axis when z is interpreted as a radius vector (Fig. 6). As in calculus, θ has an infinite number of possible values, including negative ones, that differ by integral multiples of 2π. Those values can be determined from the equation $\tan\theta = y/x$, where the quadrant containing the point corresponding to z must be specified. Each value of θ is called an *argument* of z, and the set of all such values is denoted by $\arg z$. The *principal value* of $\arg z$, denoted by $\mathrm{Arg}\ z$, is that unique value Θ such that $-\pi < \Theta \le \pi$. Evidently, then,

$$\arg z = \mathrm{Arg}\ z + 2n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots). \tag{2}$$

Also, when z is a negative real number, $\mathrm{Arg}\ z$ has value π, not $-\pi$.

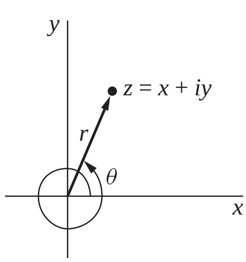

FIGURE 6

EXAMPLE 1. The complex number $-1 - i$, which lies in the third quadrant, has principal argument $-3\pi/4$. That is,

$$\mathrm{Arg}(-1 - i) = -\frac{3\pi}{4}.$$