Consequently, at points on C_R,

$$\left|\frac{z^{1/2}}{z^2+1}\right| \le M_R \quad \text{where} \quad M_R = \frac{\sqrt{R}}{R^2-1}.$$

Since the length of C_R is the number $L = \pi R$, it follows from inequality (5) that

$$\left|\int_{C_R} \frac{z^{1/2}}{z^2+1}\,dz\right| \le M_R L.$$

But

$$M_R L = \frac{\pi R\sqrt{R}}{R^2-1}\cdot\frac{1/R^2}{1/R^2} = \frac{\pi/\sqrt{R}}{1-(1/R^2)},$$

and it is clear that the term on the far right here tends to zero as R tends to infinity. Limit (7) is, therefore, established.

EXERCISES

1. Without evaluating the integral, show that

$$\left|\int_C \frac{dz}{z^2-1}\right| \le \frac{\pi}{3}$$

when C is the same arc as the one in Example 1, Sec. 43.

2. Let C denote the line segment from $z = i$ to $z = 1$. By observing that of all the points on that line segment, the midpoint is the closest to the origin, show that

$$\left|\int_C \frac{dz}{z^4}\right| \le 4\sqrt{2}$$

without evaluating the integral.

3. Show that if C is the boundary of the triangle with vertices at the points $0, 3i$, and -4, oriented in the counterclockwise direction (see Fig. 48), then

$$\left|\int_C (e^z - \overline{z})\,dz\right| \le 60.$$

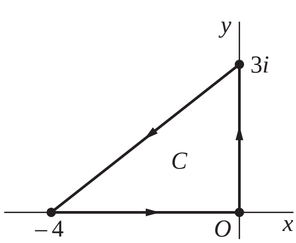

FIGURE 48

4. Let C_R denote the upper half of the circle $|z| = R\ (R > 2)$, taken in the counterclockwise direction. Show that

$$\left| \int_{C_R} \frac{2z^2 - 1}{z^4 + 5z^2 + 4}\, dz \right| \le \frac{\pi R(2R^2 + 1)}{(R^2 - 1)(R^2 - 4)}.$$

Then, by dividing the numerator and denominator on the right here by R^4, show that the value of the integral tends to zero as R tends to infinity.

5. Let C_R be the circle $|z| = R\ (R > 1)$, described in the counterclockwise direction. Show that

$$\left| \int_{C_R} \frac{\operatorname{Log} z}{z^2}\, dz \right| < 2\pi \left(\frac{\pi + \ln R}{R} \right),$$

and then use l'Hospital's rule to show that the value of this integral tends to zero as R tends to infinity.

6. Let C_ρ denote a circle $|z| = \rho\ (0 < \rho < 1)$, oriented in the counterclockwise direction, and suppose that $f(z)$ is analytic in the disk $|z| \le 1$. Show that if $z^{-1/2}$ represents any particular branch of that power of z, then there is a nonnegative constant M, *independent of* ρ, such that

$$\left| \int_{C_\rho} z^{-1/2} f(z)\, dz \right| \le 2\pi M \sqrt{\rho}.$$

Thus show that the value of the integral here approaches 0 as ρ tends to 0.

Suggestion: Note that since $f(z)$ is analytic, and therefore continuous, throughout the disk $|z| \le 1$, it is bounded there (Sec. 18).

7. Apply inequality (1), Sec. 43, to show that for all values of x in the interval $-1 \le x \le 1$, the functions*

$$P_n(x) = \frac{1}{\pi} \int_0^\pi (x + i\sqrt{1 - x^2} \cos\theta)^n\, d\theta \qquad (n = 0, 1, 2, \ldots)$$

satisfy the inequality $|P_n(x)| \le 1$.

8. Let C_N denote the boundary of the square formed by the lines

$$x = \pm \left(N + \frac{1}{2} \right) \pi \quad \text{and} \quad y = \pm \left(N + \frac{1}{2} \right) \pi,$$

where N is a positive integer and the orientation of C_N is counterclockwise.

(a) With the aid of the inequalities

$$|\sin z| \ge |\sin x| \quad \text{and} \quad |\sin z| \ge |\sinh y|,$$

obtained in Exercises 8*(a)* and 9*(a)* of Sec. 34, show that $|\sin z| \ge 1$ on the vertical sides of the square and that $|\sin z| > \sinh(\pi/2)$ on the horizontal sides. Thus show that there is a positive constant A, *independent of* N, such that $|\sin z| \ge A$ for all points z lying on the contour C_N.

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*These functions are actually polynomials in x. They are known as *Legendre polynomials* and are important in applied mathematics. See, for example, Chap. 4 of the book by Lebedev that is listed in Appendix 1.

(b) Using the final result in part *(a)*, show that

$$\left| \int_{C_N} \frac{dz}{z^2 \sin z} \right| \le \frac{16}{(2N+1)\pi A}$$

and hence that the value of this integral tends to zero as N tends to infinity.

44. ANTIDERIVATIVES

Although the value of a contour integral of a function $f(z)$ from a fixed point z_1 to a fixed point z_2 depends, in general, on the path that is taken, there are certain functions whose integrals from z_1 to z_2 have values that are *independent of path*. (Recall Examples 2 and 3 in Sec. 41.) The examples just cited also illustrate the fact that the values of integrals around closed paths are sometimes, but not always, zero. Our next theorem is useful in determining when integration is independent of path and, moreover, when an integral around a closed path has value zero.

The theorem contains an extension of the fundamental theorem of calculus that simplifies the evaluation of many contour integrals. The extension involves the concept on an antiderivative of a continuous function $f(z)$ on a domain D, or a function $F(z)$ such that $F'(z) = f(z)$ for all z in D. Note that an antiderivative is, of necessity, an analytic function. Note, too, that *an antiderivative of a given function $f(z)$ is unique except for an additive constant*. This is because the derivative of the difference $F(z) - G(z)$ of any two such antiderivatives is zero; and, according to the theorem in Sec. 24, an analytic function is constant in a domain D when its derivative is zero throughout D.

Theorem. *Suppose that a function $f(z)$ is continuous on a domain D. If any one of the following statements is true, then so are the others:*

(a) *$f(z)$ has an antiderivative $F(z)$ throughout D;*

(b) *the integrals of $f(z)$ along contours lying entirely in D and extending from any fixed point z_1 to any fixed point z_2 all have the same value, namely*

$$\int_{z_1}^{z_2} f(z)\,dz = F(z)\Big]_{z_1}^{z_2} = F(z_2) - F(z_1)$$

where $F(z)$ is the antiderivative in statement (a);

(c) *the integrals of $f(z)$ around closed contours lying entirely in D all have value zero.*

It should be emphasized that the theorem does *not* claim that any of these statements is true for a given function $f(z)$. It says only that all of them are true or that none of them is true. The next section is devoted to the proof of the theorem and can be easily skipped by a reader who wishes to get on with other important aspects of integration theory. But we include here a number of examples illustrating how the theorem can be used.