Consequently, if the point $z + \Delta z$ is close enough to z so that $|\Delta z| < \delta$, then

$$\left| \frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z) \right| < \frac{1}{|\Delta z|} \varepsilon |\Delta z| = \varepsilon;$$

that is,

$$\lim_{\Delta z \to 0} \frac{F(z + \Delta z) - F(z)}{\Delta z} = f(z),$$

or $F'(z) = f(z)$.

EXERCISES

1. Use an antiderivative to show that for every contour C extending from a point z_1 to a point z_2,

$$\int_C z^n \, dz = \frac{1}{n+1}(z_2^{n+1} - z_1^{n+1}) \qquad (n = 0, 1, 2, \ldots).$$

2. By finding an antiderivative, evaluate each of these integrals, where the path is any contour between the indicated limits of integration:

(a) $\displaystyle\int_i^{i/2} e^{\pi z} \, dz$; *(b)* $\displaystyle\int_0^{\pi + 2i} \cos\left(\frac{z}{2}\right) dz$; *(c)* $\displaystyle\int_1^3 (z - 2)^3 \, dz$.

Ans. (a) $(1 + i)/\pi$; *(b)* $e + (1/e)$; *(c)* 0.

3. Use the theorem in Sec. 44 to show that

$$\int_{C_0} (z - z_0)^{n-1} \, dz = 0 \qquad (n = \pm 1, \pm 2, \ldots)$$

when C_0 is any closed contour which does not pass through the point z_0. [Compare with Exercise 10*(b)*, Sec. 42.]

4. Find an antiderivative $F_2(z)$ of the branch $f_2(z)$ of $z^{1/2}$ in Example 4, Sec. 44, to show that integral (6) there has value $2\sqrt{3}(-1 + i)$. Note that the value of the integral of the function (5) around the closed contour $C_2 - C_1$ in that example is, therefore, $-4\sqrt{3}$.

5. Show that

$$\int_{-1}^{1} z^i \, dz = \frac{1 + e^{-\pi}}{2}(1 - i),$$

where the integrand denotes the principal branch

$$z^i = \exp(i \operatorname{Log} z) \qquad (|z| > 0, -\pi < \operatorname{Arg} z < \pi)$$

of z^i and where the path of integration is any contour from $z = -1$ to $z = 1$ that, except for its end points, lies above the real axis. (Compare with Exercise 7, Sec. 42.)

Suggestion: Use an antiderivative of the branch

$$z^i = \exp(i \log z) \qquad \left(|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right)$$

of the same power function.