If f is a constant function, then $|f(z)| = M$ for all z in R. If, however, $f(z)$ is not constant, then, according to the theorem just proved, $|f(z)| \neq M$ for any point z in the interior of R. We thus arrive at an important corollary.

Corollary. *Suppose that a function f is continuous on a closed bounded region R and that it is analytic and not constant in the interior of R. Then the maximum value of $|f(z)|$ in R, which is always reached, occurs somewhere on the boundary of R and never in the interior.*

EXAMPLE. Let R denote the rectangular region $0 \le x \le \pi$, $0 \le y \le 1$. The corollary tells us that the modulus of the entire function $f(z) = \sin z$ has a maximum value in R that occurs somewhere on the boundary of R and not in its interior. This can be verified directly by writing (see Sec. 34)

$$|f(z)| = \sqrt{\sin^2 x + \sinh^2 y}$$

and noting that the term $\sin^2 x$ is greatest when $x = \pi/2$ and that the increasing function $\sinh^2 y$ is greatest when $y = 1$. Thus the maximum value of $|f(z)|$ in R occurs at the boundary point $z = (\pi/2, 1)$ and at no other point in R (Fig. 72).

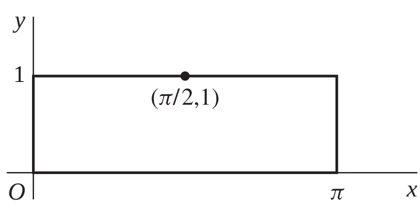

FIGURE 72

When the function f in the corollary is written $f(z) = u(x, y) + iv(x, y)$, *the component function $u(x, y)$ also has a maximum value in R which is assumed on the boundary of R and never in the interior*, where it is harmonic (Sec. 26). This is because the composite function $g(z) = \exp[f(z)]$ is continuous in R and analytic and not constant in the interior. Hence its modulus $|g(z)| = \exp[u(x, y)]$, which is continuous in R, must assume its maximum value in R on the boundary. In view of the increasing nature of the exponential function, it follows that the maximum value of $u(x, y)$ also occurs on the boundary.

Properties of *minimum* values of $|f(z)|$ and $u(x, y)$ are treated in the exercises.

EXERCISES

1. Suppose that $f(z)$ is entire and that the harmonic function $u(x, y) = \text{Re}[f(z)]$ has an upper bound u_0; that is, $u(x, y) \le u_0$ for all points (x, y) in the xy plane. Show that $u(x, y)$ must be constant throughout the plane.

Suggestion: Apply Liouville's theorem (Sec. 53) to the function $g(z) = \exp[f(z)]$.

2. Show that for R sufficiently large, the polynomial $P(z)$ in Theorem 2, Sec. 53, satisfies the inequality

$$|P(z)| < 2|a_n||z|^n \quad \text{whenever} \quad |z| \geq R.$$

[Compare with the first of inequalities (5), Sec. 53.]

Suggestion: Observe that there is a positive number R such that the modulus of each quotient in expression (3), Sec. 53, is less than $|a_n|/n$ when $|z| > R$.

3. Let a function f be continuous on a closed bounded region R, and let it be analytic and not constant throughout the interior of R. Assuming that $f(z) \neq 0$ anywhere in R, prove that $|f(z)|$ has a *minimum value* m in R which occurs on the boundary of R and never in the interior. Do this by applying the corresponding result for maximum values (Sec. 54) to the function $g(z) = 1/f(z)$.

4. Use the function $f(z) = z$ to show that in Exercise 3 the condition $f(z) \neq 0$ anywhere in R is necessary in order to obtain the result of that exercise. That is, show that $|f(z)|$ *can* reach its minimum value at an interior point when the minimum value is zero.

5. Consider the function $f(z) = (z + 1)^2$ and the closed triangular region R with vertices at the points $z = 0$, $z = 2$, and $z = i$. Find points in R where $|f(z)|$ has its maximum and minimum values, thus illustrating results in Sec. 54 and Exercise 3.

Suggestion: Interpret $|f(z)|$ as the square of the distance between z and -1.

Ans. $z = 2, z = 0$.

6. Let $f(z) = u(x, y) + iv(x, y)$ be a function that is continuous on a closed bounded region R and analytic and not constant throughout the interior of R. Prove that the component function $u(x, y)$ has a minimum value in R which occurs on the boundary of R and never in the interior. (See Exercise 3.)

7. Let f be the function $f(z) = e^z$ and R the rectangular region $0 \leq x \leq 1, 0 \leq y \leq \pi$. Illustrate results in Sec. 54 and Exercise 6 by finding points in R where the component function $u(x, y) = \text{Re}[f(z)]$ reaches its maximum and minimum values.

Ans. $z = 1, z = 1 + \pi i$.

8. Let the function $f(z) = u(x, y) + iv(x, y)$ be continuous on a closed bounded region R, and suppose that it is analytic and not constant in the interior of R. Show that the component function $v(x, y)$ has maximum and minimum values in R which are reached on the boundary of R and never in the interior, where it is harmonic.

Suggestion: Apply results in Sec. 54 and Exercise 6 to the function $g(z) = -if(z)$.

9. Let z_0 be a zero of the polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \qquad (a_n \neq 0)$$

of degree n $(n \geq 1)$. Show in the following way that

$$P(z) = (z - z_0)Q(z)$$

where $Q(z)$ is a polynomial of degree $n - 1$.

(a) Verify that

$$z^k - z_0^k = (z - z_0)(z^{k-1} + z^{k-2}z_0 + \cdots + z\,z_0^{k-2} + z_0^{k-1}) \qquad (k = 2, 3, \ldots).$$

(b) Use the factorization in part *(a)* to show that

$$P(z) - P(z_0) = (z - z_0)Q(z)$$

where $Q(z)$ is a polynomial of degree $n - 1$, and deduce the desired result from this.