EXERCISES

1. Use definition (2), Sec. 55, of limits of sequences to verify the limit of the sequence z_n $(n = 1, 2, \ldots)$ found in Example 2, Sec. 55.

2. Let Θ_n $(n = 1, 2, \ldots)$ denote the principal arguments of the numbers

$$z_n = 2 + i\frac{(-1)^n}{n^2} \qquad (n = 1, 2, \ldots).$$

Point out why

$$\lim_{n\to\infty} \Theta_n = 0,$$

and compare with Example 2, Sec. 55.

3. Use the inequality (see Sec. 4) $||z_n| - |z|| \le |z_n - z|$ to show that

$$\text{if} \quad \lim_{n\to\infty} z_n = z\,, \quad \text{then} \quad \lim_{n\to\infty} |z_n| = |z|.$$

4. Write $z = re^{i\theta}$, where $0 < r < 1$, in the summation formula (10), Sec. 56. Then, with the aid of the theorem in Sec. 56, show that

$$\sum_{n=1}^{\infty} r^n \cos n\theta = \frac{r\cos\theta - r^2}{1 - 2r\cos\theta + r^2} \quad \text{and} \quad \sum_{n=1}^{\infty} r^n \sin n\theta = \frac{r\sin\theta}{1 - 2r\cos\theta + r^2}$$

when $0 < r < 1$. (Note that these formulas are also valid when $r = 0$.)

5. Show that a limit of a convergent sequence of complex numbers is unique by appealing to the corresponding result for a sequence of real numbers.

6. Show that

$$\text{if} \quad \sum_{n=1}^{\infty} z_n = S, \quad \text{then} \quad \sum_{n=1}^{\infty} \overline{z_n} = \overline{S}.$$

7. Let c denote any complex number and show that

$$\text{if} \quad \sum_{n=1}^{\infty} z_n = S, \quad \text{then} \quad \sum_{n=1}^{\infty} cz_n = cS.$$

8. By recalling the corresponding result for series of real numbers and referring to the theorem in Sec. 56, show that

$$\text{if} \quad \sum_{n=1}^{\infty} z_n = S \quad \text{and} \quad \sum_{n=1}^{\infty} w_n = T, \qquad \text{then} \quad \sum_{n=1}^{\infty} (z_n + w_n) = S + T.$$

9. Let a sequence z_n $(n = 1, 2, \ldots)$ converge to a number z. Show that there exists a positive number M such that the inequality $|z_n| \le M$ holds for all n. Do this in each of the following ways.

(a) Note that there is a positive integer n_0 such that

$$|z_n| = |z + (z_n - z)| < |z| + 1$$

whenever $n > n_0$.

(b) Write $z_n = x_n + iy_n$ and recall from the theory of sequences of real numbers that the convergence of x_n and y_n $(n = 1, 2, \ldots)$ implies that $|x_n| \le M_1$ and $|y_n| \le M_2$ $(n = 1, 2, \ldots)$ for some positive numbers M_1 and M_2.

57. TAYLOR SERIES

We turn now to *Taylor's theorem,* which is one of the most important results of the chapter.

Theorem. *Suppose that a function f is analytic throughout a disk $|z - z_0| < R_0$, centered at z_0 and with radius R_0* (Fig. 74). *Then $f(z)$ has the power series representation*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad (|z - z_0| < R_0), \tag{1}$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!} \qquad (n = 0, 1, 2, \ldots). \tag{2}$$

That is, series (1) *converges to $f(z)$ when z lies in the stated open disk.*

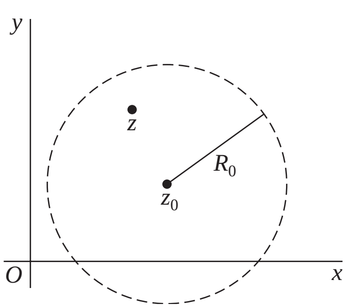

FIGURE 74

This is the expansion of $f(z)$ into a *Taylor series* about the point z_0. It is the familiar Taylor series from calculus, adapted to functions of a complex variable. With the agreement that

$$f^{(0)}(z_0) = f(z_0) \quad \text{and} \quad 0! = 1,$$

series (1) can, of course, be written

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots \qquad (|z - z_0| < R_0). \tag{3}$$