If we substitute $-z$ for z in equation (6) and its condition of validity, and note that $|z| < 1$ when $|-z| < 1$, we see that

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n \qquad (|z| < 1).$$

If, on the other hand, we replace the variable z in equation (6) by $1 - z$, we have the Taylor series representation

$$\frac{1}{z} = \sum_{n=0}^{\infty} (-1)^n (z-1)^n \qquad (|z-1| < 1).$$

This condition of validity follows from the one associated with expansion (6) since $|1 - z| < 1$ is the same as $|z - 1| < 1$.

EXAMPLE 5. For our final example, let us expand the function

$$f(z) = \frac{1+2z^2}{z^3+z^5} = \frac{1}{z^3} \cdot \frac{2(1+z^2)-1}{1+z^2} = \frac{1}{z^3}\left(2 - \frac{1}{1+z^2}\right)$$

into a series involving powers of z. We cannot find a Maclaurin series for $f(z)$ since it is not analytic at $z = 0$. But we do know from expansion (6) that

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + z^8 - \cdots \qquad (|z| < 1).$$

Hence, when $0 < |z| < 1$,

$$f(z) = \frac{1}{z^3}(2 - 1 + z^2 - z^4 + z^6 - z^8 + \cdots) = \frac{1}{z^3} + \frac{1}{z} - z + z^3 - z^5 + \cdots.$$

We call such terms as $1/z^3$ and $1/z$ *negative* powers of z since they can be written z^{-3} and z^{-1}, respectively. The theory of expansions involving negative powers of $z - z_0$ will be discussed in the next section.

EXERCISES*

1. Obtain the Maclaurin series representation

$$z\cosh(z^2) = \sum_{n=0}^{\infty} \frac{z^{4n+1}}{(2n)!} \qquad (|z| < \infty).$$

*In these and subsequent exercises on series expansions, it is recommended that the reader use, when possible, representations (1) through (6) in Sec. 59.

2. Obtain the Taylor series

$$e^z = e\sum_{n=0}^{\infty}\frac{(z-1)^n}{n!} \qquad (|z-1|<\infty)$$

for the function $f(z) = e^z$ by
(a) using $f^{(n)}(1)$ $(n = 0, 1, 2, \ldots)$; *(b)* writing $e^z = e^{z-1}e$.

3. Find the Maclaurin series expansion of the function

$$f(z) = \frac{z}{z^4+9} = \frac{z}{9}\cdot\frac{1}{1+(z^4/9)}.$$

Ans. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{3^{2n+2}}z^{4n+1} \quad (|z|<\sqrt{3}).$

4. Show that if $f(z) = \sin z$, then

$$f^{(2n)}(0) = 0 \quad \text{and} \quad f^{(2n+1)}(0) = (-1)^n \qquad (n = 0, 1, 2, \ldots).$$

Thus give an alternative derivation of the Maclaurin series (2) for $\sin z$ in Sec. 59.

5. Rederive the Maclaurin series (3) in Sec. 59 for the function $f(z) = \cos z$ by
(a) using the definition

$$\cos z = \frac{e^{iz}+e^{-iz}}{2}$$

in Sec. 34 and appealing to the Maclaurin series (1) for e^z in Sec. 59;
(b) showing that

$$f^{(2n)}(0) = (-1)^n \quad \text{and} \quad f^{(2n+1)}(0) = 0 \qquad (n = 0, 1, 2, \ldots).$$

6. Use representation (2), Sec. 59, for sin z to write the Maclaurin series for the function

$$f(z) = \sin(z^2),$$

and point out how it follows that

$$f^{(4n)}(0) = 0 \quad \text{and} \quad f^{(2n+1)}(0) = 0 \qquad (n = 0, 1, 2, \ldots).$$

7. Derive the Taylor series representation

$$\frac{1}{1-z} = \sum_{n=0}^{\infty}\frac{(z-i)^n}{(1-i)^{n+1}} \qquad (|z-i|<\sqrt{2}).$$

Suggestion: Start by writing

$$\frac{1}{1-z} = \frac{1}{(1-i)-(z-i)} = \frac{1}{1-i}\cdot\frac{1}{1-(z-i)/(1-i)}.$$

8. With the aid of the identity (see Sec. 34)

$$\cos z = -\sin\left(z - \frac{\pi}{2}\right),$$

expand $\cos z$ into a Taylor series about the point $z_0 = \pi/2$.

9. Use the identity $\sinh(z + \pi i) = -\sinh z$, verified in Exercise 7*(a)*, Sec. 35, and the fact that $\sinh z$ is periodic with period $2\pi i$ to find the Taylor series for $\sinh z$ about the point $z_0 = \pi i$.

$$\textit{Ans.}\ -\sum_{n=0}^{\infty} \frac{(z - \pi i)^{2n+1}}{(2n+1)!} \qquad (|z - \pi i| < \infty).$$

10. What is the largest circle within which the Maclaurin series for the function $\tanh z$ converges to $\tanh z$? Write the first two nonzero terms of that series.

11. Show that when $z \neq 0$,

(a) $\dfrac{e^z}{z^2} = \dfrac{1}{z^2} + \dfrac{1}{z} + \dfrac{1}{2!} + \dfrac{z}{3!} + \dfrac{z^2}{4!} + \cdots;$

(b) $\dfrac{\sin(z^2)}{z^4} = \dfrac{1}{z^2} - \dfrac{z^2}{3!} + \dfrac{z^6}{5!} - \dfrac{z^{10}}{7!} + \cdots.$

12. Derive the expansions

(a) $\dfrac{\sinh z}{z^2} = \dfrac{1}{z} + \displaystyle\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+3)!} \qquad (0 < |z| < \infty);$

(b) $z^3 \cosh\left(\dfrac{1}{z}\right) = \dfrac{z}{2} + z^3 + \displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n+2)!} \cdot \frac{1}{z^{2n-1}} \qquad (0 < |z| < \infty).$

13. Show that when $0 < |z| < 4$,

$$\frac{1}{4z - z^2} = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}.$$

60. LAURENT SERIES

If a function f fails to be analytic at a point z_0, one cannot apply Taylor's theorem at that point. It is often possible, however, to find a series representation for $f(z)$ involving both positive and negative powers of $z - z_0$. (See Example 5, Sec. 59, and also Exercises 11, 12, and 13 for that section.) We now present the theory of such representations, and we begin with *Laurent's theorem.*

Theorem. *Suppose that a function f is analytic throughout an annular domain $R_1 < |z - z_0| < R_2$, centered at z_0, and let C denote any positively oriented simple closed contour around z_0 and lying in that domain* (Fig. 76). *Then, at each point in the domain, $f(z)$ has the series representation*

$$(1) \qquad f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \qquad (R_1 < |z - z_0| < R_2),$$