If we replace the index of summation n in the first of these series by $n-1$ and then interchange the two series, we arrive at an expansion having the same form as the one in the statement of Laurent's theorem (Sec. 60):

$$f(z)=\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}+\sum_{n=1}^{\infty}\frac{1}{z^n}\quad(1<|z|<2). \tag{4}$$

Since there is only one Laurent series for $f(z)$ in the annulus D_2, expansion (4) is, in fact, *the* Laurent series for $f(z)$ there.

EXAMPLE 5. The representation of the function (1) in the unbounded domain D_3, where $2<|z|<\infty$, is also a Laurent series. Since $|2/z|<1$ when z is in D_3, it is also true that $|1/z|<1$. So if we write expression (1) as

$$f(z)=\frac{1}{2}\cdot\frac{1}{1-(1/z)}-\frac{1}{z}\cdot\frac{1}{1-(2/z)},$$

we find that

$$f(z)=\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}-\sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}=\sum_{n=0}^{\infty}\frac{1-2^n}{z^{n+1}}\quad(2<|z|<\infty).$$

Replacing n by $n-1$ in this last series then gives the standard form

$$f(z)=\sum_{n=1}^{\infty}\frac{1-2^{n-1}}{z^n}\quad(2<|z|<\infty) \tag{5}$$

used in Laurent's theorem in Sec. 60. Here, of course, all the a_n's in that theorem are zero.

EXERCISES

1. Find the Laurent series that represents the function

$$f(z)=z^2\sin\left(\frac{1}{z^2}\right)$$

in the domain $0<|z|<\infty$.

Ans. $1+\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!}\cdot\frac{1}{z^{4n}}$.

2. Derive the Laurent series representation

$$\frac{e^z}{(z+1)^2}=\frac{1}{e}\left[\sum_{n=0}^{\infty}\frac{(z+1)^n}{(n+2)!}+\frac{1}{z+1}+\frac{1}{(z+1)^2}\right]\qquad(0<|z+1|<\infty).$$

3. Find a representation for the function

$$f(z) = \frac{1}{1+z} = \frac{1}{z} \cdot \frac{1}{1+(1/z)}$$

in negative powers of z that is valid when $1 < |z| < \infty$.

Ans. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{z^n}$.

4. Give two Laurent series expansions in powers of z for the function

$$f(z) = \frac{1}{z^2(1-z)},$$

and specify the regions in which those expansions are valid.

Ans. $\displaystyle\sum_{n=0}^{\infty} z^n + \frac{1}{z} + \frac{1}{z^2} \quad (0 < |z| < 1); \qquad -\sum_{n=3}^{\infty} \frac{1}{z^n} \quad (1 < |z| < \infty)$.

5. Represent the function

$$f(z) = \frac{z+1}{z-1}$$

(a) by its Maclaurin series, and state where the representation is valid;
(b) by its Laurent series in the domain $1 < |z| < \infty$.

Ans. (a) $\displaystyle -1 - 2\sum_{n=1}^{\infty} z^n \quad (|z| < 1)$; *(b)* $\displaystyle 1 + 2\sum_{n=1}^{\infty} \frac{1}{z^n}$.

6. Show that when $0 < |z-1| < 2$,

$$\frac{z}{(z-1)(z-3)} = -3\sum_{n=0}^{\infty} \frac{(z-1)^n}{2^{n+2}} - \frac{1}{2(z-1)}.$$

7. Write the two Laurent series in powers of z that represent the function

$$f(z) = \frac{1}{z(1+z^2)}$$

in certain domains, and specify those domains.

Ans. $\displaystyle\sum_{n=0}^{\infty} (-1)^{n+1} z^{2n+1} + \frac{1}{z} \quad (0 < |z| < 1); \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{z^{2n+1}} \quad (1 < |z| < \infty)$.

8. *(a)* Let a denote a real number, where $-1 < a < 1$, and derive the Laurent series representation

$$\frac{a}{z-a} = \sum_{n=1}^{\infty} \frac{a^n}{z^n} \qquad (|a| < |z| < \infty).$$

(b) After writing $z = e^{i\theta}$ in the equation obtained in part *(a)*, equate real parts and then imaginary parts on each side of the result to derive the summation formulas

$$\sum_{n=1}^{\infty} a^n \cos n\theta = \frac{a\cos\theta - a^2}{1 - 2a\cos\theta + a^2} \quad \text{and} \quad \sum_{n=1}^{\infty} a^n \sin n\theta = \frac{a\sin\theta}{1 - 2a\cos\theta + a^2},$$

where $-1 < a < 1$. (Compare with Exercise 4, Sec. 56.)

9. Suppose that a series

$$\sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

converges to an analytic function $X(z)$ in some annulus $R_1 < |z| < R_2$. That sum $X(z)$ is called the *z-transform* of $x[n]$ $(n = 0, \pm1, \pm2, \ldots)$.* Use expression (5), Sec. 60, for the coefficients in a Laurent series to show that if the annulus contains the unit circle $|z| = 1$, then the *inverse* z-transform of $X(z)$ can be written

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{i\theta})e^{in\theta}\,d\theta \qquad (n = 0, \pm1, \pm2, \ldots).$$

10. *(a)* Let z be any complex number, and let C denote the unit circle

$$w = e^{i\phi} \qquad (-\pi \le \phi \le \pi)$$

in the w plane. Then use that contour in expression (5), Sec. 60, for the coefficients in a Laurent series, adapted to such series about the origin in the w plane, to show that

$$\exp\left[\frac{z}{2}\left(w - \frac{1}{w}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(z)w^n \qquad (0 < |w| < \infty)$$

where

$$J_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp[-i(n\phi - z\sin\phi)]\,d\phi \qquad (n = 0, \pm1, \pm2, \ldots).$$

(b) With the aid of Exercise 5, Sec. 38, regarding certain definite integrals of even and odd complex-valued functions of a real variable, show that the coefficients in part *(a)* here can be written†

$$J_n(z) = \frac{1}{\pi}\int_{0}^{\pi} \cos(n\phi - z\sin\phi)\,d\phi \qquad (n = 0, \pm1, \pm2, \ldots).$$

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*The z-transform arises in studies of discrete-time linear systems. See, for instance, the book by Oppenheim, Schafer, and Buck that is listed in Appendix 1.

†These coefficients $J_n(z)$ are called *Bessel functions* of the first kind. They play a prominent role in certain areas of applied mathematics. See, for example, the authors' "Fourier Series and Boundary Value Problems," 7th ed., Chap. 9, 2008.

11. *(a)* Let $f(z)$ denote a function which is analytic in some annular domain about the origin that includes the unit circle $z = e^{i\phi}$ $(-\pi \le \phi \le \pi)$. By taking that circle as the path of integration in expressions (2) and (3), Sec. 60, for the coefficients a_n and b_n in a Laurent series in powers of z, show that

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi})\,d\phi + \frac{1}{2\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} f(e^{i\phi})\left[\left(\frac{z}{e^{i\phi}}\right)^n + \left(\frac{e^{i\phi}}{z}\right)^n\right]d\phi$$

when z is any point in the annular domain.

(b) Write $u(\theta) = \mathrm{Re}[f(e^{i\theta})]$ and show how it follows from the expansion in part *(a)* that

$$u(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\phi)\,d\phi + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} u(\phi)\cos[n(\theta-\phi)]\,d\phi.$$

This is one form of the *Fourier series* expansion of the real-valued function $u(\theta)$ on the interval $-\pi \le \theta \le \pi$. The restriction on $u(\theta)$ is more severe than is necessary in order for it to be represented by a Fourier series.*

63. ABSOLUTE AND UNIFORM CONVERGENCE OF POWER SERIES

This section and the three following it are devoted mainly to various properties of power series. A reader who wishes to simply accept the theorems and the corollary in these sections can easily skip the proofs in order to reach Sec. 67 more quickly.

We recall from Sec. 56 that a series of complex numbers converges *absolutely* if the series of absolute values of those numbers converges. The following theorem concerns the absolute convergence of power series.

Theorem 1. *If a power series*

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n \tag{1}$$

converges when $z = z_1$ $(z_1 \ne z_0)$, *then it is absolutely convergent at each point* z *in the open disk* $|z - z_0| < R_1$ *where* $R_1 = |z_1 - z_0|$ (Fig. 79).

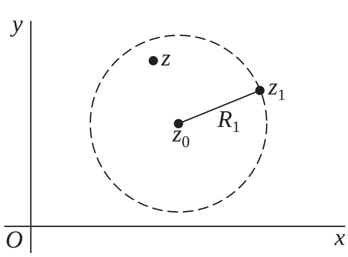

FIGURE 79

*For other sufficient conditions, see Secs. 12 and 13 of the book cited in the footnote to Exercise 10.