for each point z in it. Let $g(z)$ be as defined by equation (4), but now allow n to be a negative integer too. Also, let C be any circle around the annulus, centered at z_0 and taken in the positive sense. Then, using the index of summation m and adapting Theorem 1 in Sec. 65 to series involving both nonnegative *and* negative powers of $z - z_0$ (Exercise 10), write

$$\int_C g(z) f(z)\, dz = \sum_{m=-\infty}^{\infty} c_m \int_C g(z)(z - z_0)^m\, dz,$$

or

$$\frac{1}{2\pi i} \int_C \frac{f(z)\, dz}{(z - z_0)^{n+1}} = \sum_{m=-\infty}^{\infty} c_m \int_C g(z)(z - z_0)^m\, dz. \tag{9}$$

Since equations (6) are also valid when the integers m and n are allowed to be negative, equation (9) reduces to

$$\frac{1}{2\pi i} \int_C \frac{f(z)\, dz}{(z - z_0)^{n+1}} = c_n, \qquad (n = 0, \pm 1, \pm 2, \ldots),$$

which is expression (5), Sec. 60, for coefficients in the Laurent series for f in the annulus.

EXERCISES

1. By differentiating the Maclaurin series representation

$$\frac{1}{1 - z} = \sum_{n=0}^{\infty} z^n \qquad (|z| < 1),$$

obtain the expansions

$$\frac{1}{(1 - z)^2} = \sum_{n=0}^{\infty} (n + 1)\, z^n \qquad (|z| < 1)$$

and

$$\frac{2}{(1 - z)^3} = \sum_{n=0}^{\infty} (n + 1)(n + 2)\, z^n \qquad (|z| < 1).$$

2. By substituting $1/(1 - z)$ for z in the expansion

$$\frac{1}{(1 - z)^2} = \sum_{n=0}^{\infty} (n + 1)\, z^n \qquad (|z| < 1),$$

found in Exercise 1, derive the Laurent series representation

$$\frac{1}{z^2} = \sum_{n=2}^{\infty} \frac{(-1)^n (n - 1)}{(z - 1)^n} \qquad (1 < |z - 1| < \infty).$$

(Compare with Example 2, Sec. 65.)

3. Find the Taylor series for the function

$$\frac{1}{z} = \frac{1}{2+(z-2)} = \frac{1}{2} \cdot \frac{1}{1+(z-2)/2}$$

about the point $z_0 = 2$. Then, by differentiating that series term by term, show that

$$\frac{1}{z^2} = \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n(n+1)\left(\frac{z-2}{2}\right)^n \qquad (|z-2| < 2).$$

4. With the aid of series, show that the function f defined by means of the equations

$$f(z) = \begin{cases} (\sin z)/z & \text{when } z \neq 0, \\ 1 & \text{when } z = 0 \end{cases}$$

is entire. Use that result to establish the limit

$$\lim_{z \to 0} \frac{\sin z}{z} = 1.$$

(See Example 1, Sec. 65.)

5. Prove that if

$$f(z) = \begin{cases} \dfrac{\cos z}{z^2 - (\pi/2)^2} & \text{when } z \neq \pm\pi/2, \\ -\dfrac{1}{\pi} & \text{when } z = \pm\pi/2, \end{cases}$$

then f is an entire function.

6. In the w plane, integrate the Taylor series expansion (see Example 4, Sec. 59)

$$\frac{1}{w} = \sum_{n=0}^{\infty}(-1)^n(w-1)^n \qquad (|w-1| < 1)$$

along a contour interior to the circle of convergence from $w = 1$ to $w = z$ to obtain the representation

$$\text{Log } z = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(z-1)^n \qquad (|z-1| < 1).$$

7. Use the result in Exercise 6 to show that if

$$f(z) = \frac{\text{Log } z}{z-1} \qquad \text{when } z \neq 1$$

and $f(1) = 1$, then f is analytic throughout the domain

$$0 < |z| < \infty, \ -\pi < \text{Arg } z < \pi.$$

8. Prove that if f is analytic at z_0 and $f(z_0) = f'(z_0) = \cdots = f^{(m)}(z_0) = 0$, then the function g defined by means of the equations

$$g(z) = \begin{cases} \dfrac{f(z)}{(z-z_0)^{m+1}} & \text{when } z \neq z_0, \\ \dfrac{f^{(m+1)}(z_0)}{(m+1)!} & \text{when } z = z_0 \end{cases}$$

is analytic at z_0.

9. Suppose that a function $f(z)$ has a power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

inside some circle $|z - z_0| = R$. Use Theorem 2 in Sec. 65, regarding term by term differentiation of such a series, and mathematical induction to show that

$$f^{(n)}(z) = \sum_{k=0}^{\infty} \frac{(n+k)!}{k!} a_{n+k} (z - z_0)^k \qquad (n = 0, 1, 2, \ldots)$$

when $|z-z_0| < R$. Then, by setting $z = z_0$, show that the coefficients a_n $(n = 0, 1, 2, \ldots)$ are the coefficients in the Taylor series for f about z_0. Thus give an alternative proof of Theorem 1 in Sec. 66.

10. Consider two series

$$S_1(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad S_2(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n},$$

which converge in some annular domain centered at z_0. Let C denote any contour lying in that annulus, and let $g(z)$ be a function which is continuous on C. Modify the proof of Theorem 1, Sec. 65, which tells us that

$$\int_C g(z) S_1(z)\, dz = \sum_{n=0}^{\infty} a_n \int_C g(z)(z - z_0)^n\, dz,$$

to prove that

$$\int_C g(z) S_2(z)\, dz = \sum_{n=1}^{\infty} b_n \int_C \frac{g(z)}{(z - z_0)^n}\, dz.$$

Conclude from these results that if

$$S(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n},$$

then

$$\int_C g(z) S(z)\, dz = \sum_{n=-\infty}^{\infty} c_n \int_C g(z)(z - z_0)^n\, dz.$$

11. Show that the function

$$f_2(z) = \frac{1}{z^2 + 1} \qquad (z \neq \pm i)$$

is the analytic continuation (Sec. 27) of the function

$$f_1(z) = \sum_{n=0}^{\infty}(-1)^n z^{2n} \qquad (|z| < 1)$$

into the domain consisting of all points in the z plane except $z = \pm i$.

12. Show that the function $f_2(z) = 1/z^2$ $(z \neq 0)$ is the analytic continuation (Sec. 27) of the function

$$f_1(z) = \sum_{n=0}^{\infty}(n+1)(z+1)^n \qquad (|z+1| < 1)$$

into the domain consisting of all points in the z plane except $z = 0$.

67. MULTIPLICATION AND DIVISION OF POWER SERIES

Suppose that each of the power series

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n \quad \text{and} \quad \sum_{n=0}^{\infty} b_n(z-z_0)^n \tag{1}$$

converges within some circle $|z - z_0| = R$. Their sums $f(z)$ and $g(z)$, respectively, are then analytic functions in the disk $|z - z_0| < R$ (Sec. 65), and the product of those sums has a Taylor series expansion which is valid there:

$$f(z)g(z) = \sum_{n=0}^{\infty} c_n(z-z_0)^n \qquad (|z-z_0| < R). \tag{2}$$

According to Theorem 1 in Sec. 66, the series (1) are themselves Taylor series. Hence the first three coefficients in series (2) are given by the equations

$$c_0 = f(z_0)g(z_0) = a_0b_0,$$

$$c_1 = \frac{f(z_0)g'(z_0) + f'(z_0)g(z_0)}{1!} = a_0b_1 + a_1b_0,$$

and

$$c_2 = \frac{f(z_0)g''(z_0) + 2f'(z_0)g'(z_0) + f''(z_0)g(z_0)}{2!} = a_0b_2 + a_1b_1 + a_2b_0.$$

The general expression for any coefficient c_n is easily obtained by referring to *Leibniz's rule* (Exercise 6)

$$[f(z)g(z)]^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(z)g^{(n-k)}(z) \qquad (n = 1, 2, \ldots), \tag{3}$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad (k = 0, 1, 2, \ldots, n),$$