and, since (Sec. 7)

$$z_2^{-1} = \frac{1}{r_2} e^{-i\theta_2},$$

one can see that

$$\arg(z_2^{-1}) = -\arg z_2. \tag{3}$$

Hence

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2. \tag{4}$$

Statement (3) is, of course, to be interpreted as saying that the set of all values on the left-hand side is the same as the set of all values on the right-hand side. Statement (4) is, then, to be interpreted in the same way that statement (2) is.

EXAMPLE 2. In order to find the principal argument Arg z when

$$z = \frac{-2}{1+\sqrt{3}i},$$

observe that

$$\arg z = \arg(-2) - \arg(1+\sqrt{3}i).$$

Since

$$\operatorname{Arg}(-2) = \pi \quad \text{and} \quad \operatorname{Arg}(1+\sqrt{3}i) = \frac{\pi}{3},$$

one value of $\arg z$ is $2\pi/3$; and, because $2\pi/3$ is between $-\pi$ and π, we find that $\operatorname{Arg} z = 2\pi/3$.

EXERCISES

1. Find the principal argument $\operatorname{Arg} z$ when

(a) $z = \dfrac{i}{-2-2i}$; *(b)* $z = (\sqrt{3} - i)^6$.

Ans. (a) $-3\pi/4$; *(b)* π.

2. Show that *(a)* $|e^{i\theta}| = 1$; *(b)* $\overline{e^{i\theta}} = e^{-i\theta}$.

3. Use mathematical induction to show that

$$e^{i\theta_1} e^{i\theta_2} \cdots e^{i\theta_n} = e^{i(\theta_1+\theta_2+\cdots+\theta_n)} \qquad (n = 2, 3, \ldots).$$

4. Using the fact that the modulus $|e^{i\theta} - 1|$ is the distance between the points $e^{i\theta}$ and 1 (see Sec. 4), give a geometric argument to find a value of θ in the interval $0 \le \theta < 2\pi$ that satisfies the equation $|e^{i\theta} - 1| = 2$.

Ans. π.

5. By writing the individual factors on the left in exponential form, performing the needed operations, and finally changing back to rectangular coordinates, show that

(a) $i(1-\sqrt{3}i)(\sqrt{3}+i) = 2(1+\sqrt{3}i)$; *(b)* $5i/(2+i) = 1+2i$;

(c) $(-1+i)^7 = -8(1+i)$; *(d)* $(1+\sqrt{3}i)^{-10} = 2^{-11}(-1+\sqrt{3}i)$.

6. Show that if $\operatorname{Re} z_1 > 0$ and $\operatorname{Re} z_2 > 0$, then

$$\operatorname{Arg}(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2,$$

where principal arguments are used.

7. Let z be a nonzero complex number and n a negative integer $(n = -1, -2, \ldots)$. Also, write $z = re^{i\theta}$ and $m = -n = 1, 2, \ldots$. Using the expressions

$$z^m = r^m e^{im\theta} \quad \text{and} \quad z^{-1} = \left(\frac{1}{r}\right) e^{i(-\theta)},$$

verify that $(z^m)^{-1} = (z^{-1})^m$ and hence that the definition $z^n = (z^{-1})^m$ in Sec. 7 could have been written alternatively as $z^n = (z^m)^{-1}$.

8. Prove that two nonzero complex numbers z_1 and z_2 have the same moduli if and only if there are complex numbers c_1 and c_2 such that $z_1 = c_1 c_2$ and $z_2 = c_1 \overline{c_2}$.

Suggestion: Note that

$$\exp\left(i\frac{\theta_1+\theta_2}{2}\right)\exp\left(i\frac{\theta_1-\theta_2}{2}\right) = \exp(i\theta_1)$$

and [see Exercise 2*(b)*]

$$\exp\left(i\frac{\theta_1+\theta_2}{2}\right)\overline{\exp\left(i\frac{\theta_1-\theta_2}{2}\right)} = \exp(i\theta_2).$$

9. Establish the identity

$$1 + z + z^2 + \cdots + z^n = \frac{1-z^{n+1}}{1-z} \qquad (z \neq 1)$$

and then use it to derive *Lagrange's trigonometric identity:*

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin[(2n+1)\theta/2]}{2\sin(\theta/2)} \qquad (0 < \theta < 2\pi).$$

Suggestion: As for the first identity, write $S = 1 + z + z^2 + \cdots + z^n$ and consider the difference $S - zS$. To derive the second identity, write $z = e^{i\theta}$ in the first one.

10. Use de Moivre's formula (Sec. 7) to derive the following trigonometric identities:

(a) $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$; *(b)* $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$.

11. *(a)* Use the binomial formula (Sec. 3) and de Moivre's formula (Sec. 7) to write

$$\cos n\theta + i \sin n\theta = \sum_{k=0}^{n} \binom{n}{k} \cos^{n-k}\theta \,(i \sin\theta)^k \qquad (n = 0, 1, 2, \ldots).$$

Then define the integer m by means of the equations

$$m = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (n-1)/2 & \text{if } n \text{is odd} \end{cases}$$

and use the above summation to show that [compare with Exercise 10*(a)*]

$$\cos n\theta = \sum_{k=0}^{m} \binom{n}{2k} (-1)^k \cos^{n-2k}\theta \sin^{2k}\theta \qquad (n = 0, 1, 2, \ldots).$$

(b) Write $x = \cos\theta$ in the final summation in part *(a)* to show that it becomes a polynomial

$$T_n(x) = \sum_{k=0}^{m} \binom{n}{2k} (-1)^k x^{n-2k} (1 - x^2)^k$$

of degree n $(n = 0, 1, 2, \ldots)$ in the variable x.*

9. ROOTS OF COMPLEX NUMBERS

Consider now a point $z = re^{i\theta}$, lying on a circle centered at the origin with radius r (Fig. 10). As θ is increased, z moves around the circle in the counterclockwise direction. In particular, when θ is increased by 2π, we arrive at the original point; and the same is true when θ is decreased by 2π. It is, therefore, evident from Fig. 10 that *two nonzero complex numbers*

$$z_1 = r_1 e^{i\theta_1} \quad \textit{and} \quad z_2 = r_2 e^{i\theta_2}$$

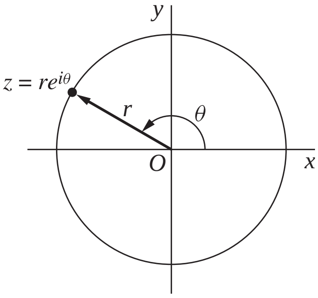

FIGURE 10

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*These are called Chebyshev polynomials and are prominent in approximation theory.