That is,

$$\frac{1}{1+z^2/3!+z^4/5!+\cdots} = 1-\frac{1}{3!}z^2+\left[\frac{1}{(3!)^2}-\frac{1}{5!}\right]z^4+\cdots,$$

or

$$\frac{1}{1+z^2/3!+z^4/5!+\cdots} = 1-\frac{1}{6}z^2+\frac{7}{360}z^4+\cdots \qquad (|z|<\pi). \tag{8}$$

Hence

$$\frac{1}{z^2\sinh z} = \frac{1}{z^3}-\frac{1}{6}\cdot\frac{1}{z}+\frac{7}{360}z+\cdots \qquad (0<|z|<\pi). \tag{9}$$

Although we have given only the first three nonzero terms of this Laurent series, any number of terms can, of course, be found by continuing the division.

EXERCISES

1. Use multiplication of series to show that

$$\frac{e^z}{z(z^2+1)} = \frac{1}{z}+1-\frac{1}{2}z-\frac{5}{6}z^2+\cdots \qquad (0<|z|<1).$$

2. By writing $\csc z = 1/\sin z$ and then using division, show that

$$\csc z = \frac{1}{z}+\frac{1}{3!}z+\left[\frac{1}{(3!)^2}-\frac{1}{5!}\right]z^3+\cdots \qquad (0<|z|<\pi).$$

3. Use division to obtain the Laurent series representation

$$\frac{1}{e^z-1} = \frac{1}{z}-\frac{1}{2}+\frac{1}{12}z-\frac{1}{720}z^3+\cdots \qquad (0<|z|<2\pi).$$

4. Use the expansion

$$\frac{1}{z^2\sinh z} = \frac{1}{z^3}-\frac{1}{6}\cdot\frac{1}{z}+\frac{7}{360}z+\cdots \qquad (0<|z|<\pi)$$

in Example 2, Sec. 67, and the method illustrated in Example 1, Sec. 62, to show that

$$\int_C \frac{dz}{z^2\sinh z} = -\frac{\pi i}{3},$$

when C is the positively oriented unit circle $|z|=1$.

5. Follow these steps, which illustrate an alternative to straightforward division, to obtain representation (8) in Example 2, Sec. 67.

(a) Write

$$\frac{1}{1+z^2/3!+z^4/5!+\cdots} = d_0+d_1z+d_2z^2+d_3z^3+d_4z^4+\cdots,$$

where the coefficients in the power series on the right are to be determined by multiplying the two series in the equation

$$1 = \left(1 + \frac{1}{3!}z^2 + \frac{1}{5!}z^4 + \cdots\right)(d_0 + d_1 z + d_2 z^2 + d_3 z^3 + d_4 z^4 + \cdots).$$

Perform this multiplication to show that

$$(d_0 - 1) + d_1 z + \left(d_2 + \frac{1}{3!}d_0\right)z^2 + \left(d_3 + \frac{1}{3!}d_1\right)z^3 + \left(d_4 + \frac{1}{3!}d_2 + \frac{1}{5!}d_0\right)z^4 + \cdots = 0$$

when $|z| < \pi$.

(b) By setting the coefficients in the last series in part *(a)* equal to zero, find the values of d_0, d_1, d_2, d_3, and d_4. With these values, the first equation in part *(a)* becomes equation (8), Sec. 67.

6. Use mathematical induction to establish Leibniz' rule (Sec. 67)

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)} \qquad (n = 1, 2, \ldots)$$

for the n^{th} derivative of the product of two differentiable functions $f(z)$ and $g(z)$.

Suggestion: Note that the rule is valid when $n = 1$. Then, assuming that it is valid when $n = m$ where m is any positive integer, show that

$$\begin{aligned}(fg)^{(m+1)} &= (fg')^{(m)} + (f'g)^{(m)} \\ &= fg^{(m+1)} + \sum_{k=1}^{m}\left[\binom{m}{k} + \binom{m}{k-1}\right] f^{(k)} g^{(m+1-k)} + f^{(m+1)}g.\end{aligned}$$

Finally, with the aid of the identify

$$\binom{m}{k} + \binom{m}{k-1} = \binom{m+1}{k}$$

that was used in Exercise 8, Sec. 3, show that

$$\begin{aligned}(fg)^{(m+1)} &= fg^{(m+1)} + \sum_{k=1}^{m}\binom{m+1}{k} f^{(k)} g^{(m+1-k)} + f^{(m+1)}g \\ &= \sum_{k=0}^{m+1}\binom{m+1}{k} f^{(k)} g^{(m+1-k)}.\end{aligned}$$

7. Let $f(z)$ be an entire function that is represented by a series of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \qquad (|z| < \infty).$$

(a) By differentiating the composite function $g(z) = f[f(z)]$ successively, find the first three nonzero terms in the Maclaurin series for $g(z)$ and thus show that

$$f[f(z)] = z + 2\,a_2 z^2 + 2\,(a_2^2 + a_3)z^3 + \cdots \qquad (|z| < \infty).$$

(b) Obtain the result in part *(a)* in a *formal* manner by writing

$$f[f(z)] = f(z) + a_2[f(z)]^2 + a_3[f(z)]^3 + \cdots,$$

replacing $f(z)$ on the right-hand side here by its series representation, and then collecting terms in like powers of z.

(c) By applying the result in part *(a)* to the function $f(z) = \sin z$, show that

$$\sin(\sin z) = z - \frac{1}{3}z^3 + \cdots \qquad (|z| < \infty).$$

8. The *Euler numbers* are the numbers E_n $(n = 0, 1, 2, \ldots)$ in the Maclaurin series representation

$$\frac{1}{\cosh z} = \sum_{n=0}^{\infty} \frac{E_n}{n!} z^n \qquad (|z| < \pi/2).$$

Point out why this representation is valid in the indicated disk and why

$$E_{2n+1} = 0 \qquad (n = 0, 1, 2, \ldots).$$

Then show that

$$E_0 = 1, \quad E_2 = -1, \quad E_4 = 5, \quad \text{and} \quad E_6 = -61.$$