**EXAMPLE.** In the example in Sec. 70, we evaluated the integral of

$$f(z) = \frac{5z-2}{z(z-1)}$$

around the circle $|z| = 2$, described counterclockwise, by finding the residues of $f(z)$ at $z = 0$ and $z = 1$. Since

$$\begin{aligned}\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{5-2z}{z(1-z)} &= \frac{5-2z}{z} \cdot \frac{1}{1-z} \\ &= \left(\frac{5}{z} - 2\right)(1 + z + z^2 + \cdots) \\ &= \frac{5}{z} + 3 + 3z + \cdots \qquad (0 < |z| < 1),\end{aligned}$$

we see that the theorem here can also be used, where the desired residue is 5. More precisely,

$$\int_C \frac{5z-2}{z(z-1)}\, dz = 2\pi i(5) = 10\pi i,$$

where $C$ is the circle in question. This is, of course, the result obtained in the example in Sec. 70.

## EXERCISES

**1.** Find the residue at $z = 0$ of the function

*(a)* $\dfrac{1}{z+z^2}$; *(b)* $z\cos\left(\dfrac{1}{z}\right)$; *(c)* $\dfrac{z - \sin z}{z}$; *(d)* $\dfrac{\cot z}{z^4}$; *(e)* $\dfrac{\sinh z}{z^4(1-z^2)}$.

*Ans.* *(a)* 1; *(b)* $-1/2$; *(c)* 0; *(d)* $-1/45$; *(e)* $7/6$.

**2.** Use Cauchy's residue theorem (Sec. 70) to evaluate the integral of each of these functions around the circle $|z| = 3$ in the positive sense:

*(a)* $\dfrac{\exp(-z)}{z^2}$; *(b)* $\dfrac{\exp(-z)}{(z-1)^2}$; *(c)* $z^2 \exp\left(\dfrac{1}{z}\right)$; *(d)* $\dfrac{z+1}{z^2-2z}$.

*Ans.* *(a)* $-2\pi i$; *(b)* $-2\pi i/e$; *(c)* $\pi i/3$; *(d)* $2\pi i$.

**3.** Use the theorem in Sec. 71, involving a single residue, to evaluate the integral of each of these functions around the circle $|z| = 2$ in the positive sense:

*(a)* $\dfrac{z^5}{1-z^3}$; *(b)* $\dfrac{1}{1+z^2}$; *(c)* $\dfrac{1}{z}$.

*Ans.* *(a)* $-2\pi i$; *(b)* 0; *(c)* $2\pi i$.

**4.** Let $C$ denote the circle $|z| = 1$, taken counterclockwise, and use the following steps to show that

$$\int_C \exp\left(z + \frac{1}{z}\right) dz = 2\pi i \sum_{n=0}^{\infty} \frac{1}{n!\,(n+1)!}.$$

*(a)* By using the Maclaurin series for $e^z$ and referring to Theorem 1 in Sec. 65, which justifies the term by term integration that is to be used, write the above integral as

$$\sum_{n=0}^{\infty} \frac{1}{n!} \int_C z^n \exp\left(\frac{1}{z}\right) dz.$$

*(b)* Apply the theorem in Sec. 70 to evaluate the integrals appearing in part *(a)* to arrive at the desired result.

**5.** Suppose that a function $f$ is analytic throughout the finite plane except for a finite number of singular points $z_1, z_2, \ldots, z_n$. Show that

$$\operatorname*{Res}_{z=z_1} f(z) + \operatorname*{Res}_{z=z_2} f(z) + \cdots + \operatorname*{Res}_{z=z_n} f(z) + \operatorname*{Res}_{z=\infty} f(z) = 0.$$

**6.** Let the degrees of the polynomials

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \qquad (a_n \neq 0)$$

and

$$Q(z) = b_0 + b_1 z + b_2 z^2 + \cdots + b_m z^m \qquad (b_m \neq 0)$$

be such that $m \geq n + 2$. Use the theorem in Sec. 71 to show that if all of the zeros of $Q(z)$ are interior to a simple closed contour $C$, then

$$\int_C \frac{P(z)}{Q(z)}\, dz = 0.$$

[Compare with Exercise 3*(b)*.]

## 72. THE THREE TYPES OF ISOLATED SINGULAR POINTS

We saw in Sec. 69 that the theory of residues is based on the fact that if $f$ has an isolated singular point at $z_0$, then $f(z)$ has a Laurent series representation

$$(1) \qquad f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots + \frac{b_n}{(z - z_0)^n} + \cdots$$

in a punctured disk $0 < |z - z_0| < R_2$. The portion

$$(2) \qquad \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots + \frac{b_n}{(z - z_0)^n} + \cdots$$

of the series, involving negative powers of $z - z_0$, is called the *principal part* of $f$ at $z_0$. We now use the principal part to identify the isolated singular point $z_0$ as one of three special types. This classification will aid us in the development of residue theory that appears in following sections.

If the principal part of $f$ at $z_0$ contains at least one nonzero term but the number of such terms is only finite, then there exists a positive integer $m$ $(m \geq 1)$ such that

$$b_m \neq 0 \quad \text{and} \quad b_{m+1} = b_{m+2} = \cdots = 0.$$