In the remaining sections of this chapter, we shall develop in greater depth the theory of the three types of isolated singular points just described. The emphasis will be on useful and efficient methods for identifying poles and finding the corresponding residues.

EXERCISES

1. In each case, write the principal part of the function at its isolated singular point and determine whether that point is a pole, a removable singular point, or an essential singular point:

(a) $z \exp\left(\dfrac{1}{z}\right)$; *(b)* $\dfrac{z^2}{1+z}$; *(c)* $\dfrac{\sin z}{z}$; *(d)* $\dfrac{\cos z}{z}$; *(e)* $\dfrac{1}{(2-z)^3}$.

2. Show that the singular point of each of the following functions is a pole. Determine the order m of that pole and the corresponding residue B.

(a) $\dfrac{1-\cosh z}{z^3}$; *(b)* $\dfrac{1-\exp(2z)}{z^4}$; *(c)* $\dfrac{\exp(2z)}{(z-1)^2}$.

Ans. (a) $m = 1, B = -1/2$; *(b)* $m = 3, B = -4/3$; *(c)* $m = 2, B = 2e^2$.

3. Suppose that a function f is analytic at z_0, and write $g(z) = f(z)/(z - z_0)$. Show that

(a) if $f(z_0) \neq 0$, then z_0 is a simple pole of g, with residue $f(z_0)$;

(b) if $f(z_0) = 0$, then z_0 is a removable singular point of g.

Suggestion: As pointed out in Sec. 57, there is a Taylor series for $f(z)$ about z_0 since f is analytic there. Start each part of this exercise by writing out a few terms of that series.

4. Use the fact (see Sec. 29) that $e^z = -1$ when

$$z = (2n+1)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$$

to show that $e^{1/z}$ assumes the value -1 an infinite number of times in each neighborhood of the origin. More precisely, show that $e^{1/z} = -1$ when

$$z = -\frac{i}{(2n+1)\pi} \quad (n = 0, \pm 1, \pm 2, \ldots);$$

then note that if n is large enough, such points lie in any given ε neighborhood of the origin. Zero is evidently the exceptional value in Picard's theorem, stated in Example 5, Sec. 72.

5. Write the function

$$f(z) = \frac{8a^3 z^2}{(z^2+a^2)^3} \qquad (a > 0)$$

as

$$f(z) = \frac{\phi(z)}{(z-ai)^3} \quad \text{where} \quad \phi(z) = \frac{8a^3 z^2}{(z+ai)^3}.$$

Point out why $\phi(z)$ has a Taylor series representation about $z = ai$, and then use it to show that the principal part of f at that point is

$$\frac{\phi''(ai)/2}{z-ai} + \frac{\phi'(ai)}{(z-ai)^2} + \frac{\phi(ai)}{(z-ai)^3} = -\frac{i/2}{z-ai} - \frac{a/2}{(z-ai)^2} - \frac{a^2 i}{(z-ai)^3}.$$

73. RESIDUES AT POLES

When a function f has an isolated singularity at a point z_0, the basic method for identifying z_0 as a pole and finding the residue there is to write the appropriate Laurent series and to note the coefficient of $1/(z - z_0)$. The following theorem provides an alternative characterization of poles and a way of finding residues at poles that is often more convenient.

Theorem. *An isolated singular point z_0 of a function f is a pole of order m if and only if $f(z)$ can be written in the form*

$$f(z) = \frac{\phi(z)}{(z-z_0)^m}, \tag{1}$$

where $\phi(z)$ is analytic and nonzero at z_0. Moreover,

$$\operatorname*{Res}_{z=z_0} f(z) = \phi(z_0) \quad \text{if } m = 1 \tag{2}$$

and

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!} \quad \text{if } m \geq 2. \tag{3}$$

Observe that expression (2) need not have been written separately since, with the convention that $\phi^{(0)}(z_0) = \phi(z_0)$ and $0! = 1$, expression (3) reduces to it when $m = 1$.

To prove the theorem, we first assume that $f(z)$ has the form (1) and recall (Sec. 57) that since $\phi(z)$ is analytic at z_0, it has a Taylor series representation

$$\phi(z) = \phi(z_0) + \frac{\phi'(z_0)}{1!}(z-z_0) + \frac{\phi''(z_0)}{2!}(z-z_0)^2 + \cdots + \frac{\phi^{(m-1)}(z_0)}{(m-1)!}(z-z_0)^{m-1} + \sum_{n=m}^{\infty} \frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^n$$

in some neighborhood $|z - z_0| < \varepsilon$ of z_0; and from expression (1) it follows that

$$f(z) = \frac{\phi(z_0)}{(z-z_0)^m} + \frac{\phi'(z_0)/1!}{(z-z_0)^{m-1}} + \frac{\phi''(z_0)/2!}{(z-z_0)^{m-2}} + \cdots + \frac{\phi^{(m-1)}(z_0)/(m-1)!}{z-z_0} + \sum_{n=m}^{\infty} \frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^{n-m}$$