EXERCISES

1. In each case, show that any singular point of the function is a pole. Determine the order m of each pole, and find the corresponding residue B.

(a) $\dfrac{z^2+2}{z-1}$; *(b)* $\left(\dfrac{z}{2z+1}\right)^3$; *(c)* $\dfrac{\exp z}{z^2+\pi^2}$.

Ans. *(a)* $m = 1$, $B = 3$; *(b)* $m = 3$, $B = -3/16$; *(c)* $m = 1$, $B = \pm i/2\pi$.

2. Show that

(a) $\displaystyle\operatorname*{Res}_{z=-1} \frac{z^{1/4}}{z+1} = \frac{1+i}{\sqrt{2}}$ $\quad(|z| > 0, 0 < \arg z < 2\pi)$;

(b) $\displaystyle\operatorname*{Res}_{z=i} \frac{\operatorname{Log} z}{(z^2+1)^2} = \frac{\pi+2i}{8}$;

(c) $\displaystyle\operatorname*{Res}_{z=i} \frac{z^{1/2}}{(z^2+1)^2} = \frac{1-i}{8\sqrt{2}}$ $\quad(|z| > 0, 0 < \arg z < 2\pi)$.

3. Find the value of the integral

$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)}\, dz,$$

taken counterclockwise around the circle *(a)* $|z-2| = 2$; *(b)* $|z| = 4$.

Ans. *(a)* πi; *(b)* $6\pi i$.

4. Find the value of the integral

$$\int_C \frac{dz}{z^3(z+4)},$$

taken counterclockwise around the circle *(a)* $|z| = 2$; *(b)* $|z+2| = 3$.

Ans. *(a)* $\pi i/32$; *(b)* 0.

5. Evaluate the integral

$$\int_C \frac{\cosh \pi z}{z(z^2+1)}\, dz$$

when C is the circle $|z| = 2$, described in the positive sense.

Ans. $4\pi i$.

6. Use the theorem in Sec. 71, involving a single residue, to evaluate the integral of $f(z)$ around the positively oriented circle $|z| = 3$ when

(a) $f(z) = \dfrac{(3z+2)^2}{z(z-1)(2z+5)}$; *(b)* $f(z) = \dfrac{z^3(1-3z)}{(1+z)(1+2z^4)}$; *(c)* $f(z) = \dfrac{z^3 e^{1/z}}{1+z^3}$.

Ans. *(a)* $9\pi i$; *(b)* $-3\pi i$; *(c)* $2\pi i$.

7. Let z_0 be an isolated singular point of a function f and suppose that

$$f(z) = \frac{\phi(z)}{(z-z_0)^m},$$

where m is a positive integer and $\phi(z)$ is analytic and nonzero at z_0. By applying the extended form (6), Sec. 51, of the Cauchy integral formula to the function $\phi(z)$, show that

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!},$$

as stated in the theorem of Sec. 73.

Suggestion: Since there is a neighborhood $|z - z_0| < \varepsilon$ throughout which $\phi(z)$ is analytic (see Sec. 24), the contour used in the extended Cauchy integral formula can be the positively oriented circle $|z - z_0| = \varepsilon/2$.

75. ZEROS OF ANALYTIC FUNCTIONS

Zeros and poles of functions are closely related. In fact, we shall see in the next section how zeros can be a source of poles. We need, however, some preliminary results regarding zeros of analytic functions.

Suppose that a function f is analytic at a point z_0. We know from Sec. 52 that all of the derivatives $f^{(n)}(z)$ $(n = 1, 2, \ldots)$ exist at z_0. If $f(z_0) = 0$ and if there is a positive integer m such that $f^{(m)}(z_0) \neq 0$ and each derivative of lower order vanishes at z_0, then f is said to have a *zero of order* m at z_0. Our first theorem here provides a useful alternative characterization of zeros of order m.

Theorem 1. *Let a function f be analytic at a point z_0. It has a zero of order m at z_0 if and only if there is a function g, which is analytic and nonzero at z_0, such that*

$$f(z) = (z - z_0)^m g(z). \tag{1}$$

Both parts of the proof that follows use the fact (Sec. 57) that if a function is analytic at a point z_0, then it must have a Taylor series representation in powers of $z - z_0$ which is valid throughout a neighborhood $|z - z_0| < \varepsilon$ of z_0.

We start the first part of the proof by assuming that expression (1) holds and noting that since $g(z)$ is analytic at z_0, it has a Taylor series representation

$$g(z) = g(z_0) + \frac{g'(z_0)}{1!}(z - z_0) + \frac{g''(z_0)}{2!}(z - z_0)^2 + \cdots$$

in some neighborhood $|z - z_0| < \varepsilon$ of z_0. Expression (1) thus takes the form

$$f(z) = g(z_0)(z - z_0)^m + \frac{g'(z_0)}{1!}(z - z_0)^{m+1} + \frac{g''(z_0)}{2!}(z - z_0)^{m+2} + \cdots$$

when $|z - z_0| < \varepsilon$. Since this is actually a Taylor series expansion for $f(z)$, according to Theorem 1 in Sec. 66, it follows that

$$f(z_0) = f'(z_0) = f''(z_0) = \cdots = f^{(m-1)}(z_0) = 0 \tag{2}$$