and hence that z_0 is a simple pole of f. The residue there is, moreover,

$$B_0 = \frac{p(z_0)}{q'(z_0)} = \frac{z_0}{4z_0^3} = \frac{1}{4z_0^2} = \frac{1}{8i} = -\frac{i}{8}.$$

Although this residue can also be found by the method in Sec. 73, the computation is somewhat more involved.

There are formulas similar to formula (2) for residues at poles of higher order, but they are lengthier and, in general, not practical.

EXERCISES

1. Show that the point $z = 0$ is a simple pole of the function

$$f(z) = \csc z = \frac{1}{\sin z}$$

and that the residue there is unity by appealing to

(a) Theorem 2 in Sec. 76;

(b) the Laurent series for $\csc z$ that was found in Exercise 2, Sec. 67.

2. Show that

(a) $\displaystyle \operatorname*{Res}_{z=\pi i} \frac{z - \sinh z}{z^2 \sinh z} = \frac{i}{\pi}$;

(b) $\displaystyle \operatorname*{Res}_{z=\pi i} \frac{\exp(zt)}{\sinh z} + \operatorname*{Res}_{z=-\pi i} \frac{\exp(zt)}{\sinh z} = -2\cos(\pi t)$.

3. Show that

(a) $\displaystyle \operatorname*{Res}_{z=z_n}(z \sec z) = (-1)^{n+1} z_n$ where $z_n = \dfrac{\pi}{2} + n\pi \quad (n = 0, \pm 1, \pm 2, \ldots)$;

(b) $\displaystyle \operatorname*{Res}_{z=z_n}(\tanh z) = 1$ where $z_n = \left(\dfrac{\pi}{2} + n\pi\right) i \quad (n = 0, \pm 1, \pm 2, \ldots)$.

4. Let C denote the positively oriented circle $|z| = 2$ and evaluate the integral

(a) $\displaystyle \int_C \tan z\, dz$; *(b)* $\displaystyle \int_C \frac{dz}{\sinh 2z}$.

Ans. (a) $-4\pi i$; *(b)* $-\pi i$.

5. Let C_N denote the positively oriented boundary of the square whose edges lie along the lines

$$x = \pm\left(N + \frac{1}{2}\right)\pi \quad \text{and} \quad y = \pm\left(N + \frac{1}{2}\right)\pi,$$

where N is a positive integer. Show that

$$\int_{C_N} \frac{dz}{z^2 \sin z} = 2\pi i \left[\frac{1}{6} + 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2}\right].$$

Then, using the fact that the value of this integral tends to zero as N tends to infinity (Exercise 8, Sec. 43), point out how it follows that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

6. Show that

$$\int_C \frac{dz}{(z^2-1)^2+3} = \frac{\pi}{2\sqrt{2}},$$

where C is the positively oriented boundary of the rectangle whose sides lie along the lines $x = \pm 2$, $y = 0$, and $y = 1$.

Suggestion: By observing that the four zeros of the polynomial $q(z) = (z^2-1)^2+3$ are the square roots of the numbers $1 \pm \sqrt{3}i$, show that the reciprocal $1/q(z)$ is analytic inside and on C except at the points

$$z_0 = \frac{\sqrt{3}+i}{\sqrt{2}} \quad \text{and} \quad -\overline{z_0} = \frac{-\sqrt{3}+i}{\sqrt{2}}.$$

Then apply Theorem 2 in Sec. 76.

7. Consider the function

$$f(z) = \frac{1}{[q(z)]^2}$$

where q is analytic at z_0, $q(z_0) = 0$, and $q'(z_0) \neq 0$. Show that z_0 is a pole of order $m = 2$ of the function f, with residue

$$B_0 = -\frac{q''(z_0)}{[q'(z_0)]^3}.$$

Suggestion: Note that z_0 is a zero of order $m = 1$ of the function q, so that

$$q(z) = (z - z_0)g(z)$$

where $g(z)$ is analytic and nonzero at z_0. Then write

$$f(z) = \frac{\phi(z)}{(z-z_0)^2} \quad \text{where} \quad \phi(z) = \frac{1}{[g(z)]^2}.$$

The desired form of the residue $B_0 = \phi'(z_0)$ can be obtained by showing that

$$q'(z_0) = g(z_0) \quad \text{and} \quad q''(z_0) = 2g'(z_0).$$

8. Use the result in Exercise 7 to find the residue at $z = 0$ of the function

(a) $f(z) = \csc^2 z$; *(b)* $f(z) = \dfrac{1}{(z+z^2)^2}$.

Ans. (a) 0; *(b)* -2.

9. Let p and q denote functions that are analytic at a point z_0, where $p(z_0) \neq 0$ and $q(z_0) = 0$. Show that if the quotient $p(z)/q(z)$ has a pole of order m at z_0, then z_0 is a zero of order m of q. (Compare with Theorem 1 in Sec. 76.)

Suggestion: Note that the theorem in Sec. 73 enables one to write

$$\frac{p(z)}{q(z)} = \frac{\phi(z)}{(z - z_0)^m},$$

where $\phi(z)$ is analytic and nonzero at z_0. Then solve for $q(z)$.

10. Recall (Sec. 11) that a point z_0 is an accumulation point of a set S if each deleted neighborhood of z_0 contains at least one point of S. One form of the *Bolzano–Weierstrass theorem* can be stated as follows: *an infinite set of points lying in a closed bounded region R has at least one accumulation point in R*.* Use that theorem and Theorem 2 in Sec. 75 to show that if a function f is analytic in the region R consisting of all points inside and on a simple closed contour C, except possibly for poles inside C, and if all the zeros of f in R are interior to C and are of finite order, then those zeros must be finite in number.

11. Let R denote the region consisting of all points inside and on a simple closed contour C. Use the Bolzano–Weierstrass theorem (see Exercise 10) and the fact that poles are isolated singular points to show that if f is analytic in the region R except for poles interior to C, then those poles must be finite in number.

77. BEHAVIOR OF FUNCTIONS NEAR ISOLATED SINGULAR POINTS

As already indicated in Sec. 72, the behavior of a function f near an isolated singular point z_0 varies, depending on whether z_0 is a pole, a removable singular point, or an essential singular point. In this section, we develop the differences in behavior somewhat further. Since the results presented here will not be used elsewhere in the book, the reader who wishes to reach applications of residue theory more quickly may pass directly to Chap. 7 without disruption.

Theorem 1. *If z_0 is a pole of a function f, then*

$$\lim_{z \to z_0} f(z) = \infty. \tag{1}$$

To verify limit (1), we assume that f has a pole of order m at z_0 and use the theorem in Sec. 73. It tells us that

$$f(z) = \frac{\phi(z)}{(z - z_0)^m},$$

where $\phi(z)$ is analytic and nonzero at z_0. Since

$$\lim_{z \to z_0} \frac{1}{f(z)} = \lim_{z \to z_0} \frac{(z - z_0)^m}{\phi(z)} = \frac{\lim\limits_{z \to z_0} (z - z_0)^m}{\lim\limits_{z \to z_0} \phi(z)} = \frac{0}{\phi(z_0)} = 0,$$

*See, for example, A. E. Taylor and W. R. Mann. "Advanced Calculus," 3d ed., pp. 517 and 521, 1983.