It now follows from equation (2) that

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{x^2}{x^6+1}\,dx = \frac{\pi}{3},$$

or

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{x^2}{x^6+1}\,dx = \frac{\pi}{3}.$$

Since the integrand here is even, we know from equation (7) in Sec. 78 that

$$\int_{0}^{\infty} \frac{x^2}{x^6+1}\,dx = \frac{\pi}{6}. \tag{4}$$

EXERCISES

Use residues to evaluate the improper integrals in Exercises 1 through 5.

1. $\displaystyle\int_0^\infty \frac{dx}{x^2+1}$.
Ans. $\pi/2$.

2. $\displaystyle\int_0^\infty \frac{dx}{(x^2+1)^2}$.
Ans. $\pi/4$.

3. $\displaystyle\int_0^\infty \frac{dx}{x^4+1}$.
Ans. $\pi/(2\sqrt{2})$.

4. $\displaystyle\int_0^\infty \frac{x^2\,dx}{(x^2+1)(x^2+4)}$.
Ans. $\pi/6$.

5. $\displaystyle\int_0^\infty \frac{x^2\,dx}{(x^2+9)(x^2+4)^2}$.
Ans. $\pi/200$.

Use residues to find the Cauchy principal values of the integrals in Exercises 6 and 7.

6. $\displaystyle\int_{-\infty}^\infty \frac{dx}{x^2+2x+2}$.

7. $\displaystyle\int_{-\infty}^\infty \frac{x\,dx}{(x^2+1)(x^2+2x+2)}$.
Ans. $-\pi/5$.

8. Use a residue and the contour shown in Fig. 95, where $R > 1$, to establish the integration formula

$$\int_0^\infty \frac{dx}{x^3+1} = \frac{2\pi}{3\sqrt{3}}.$$

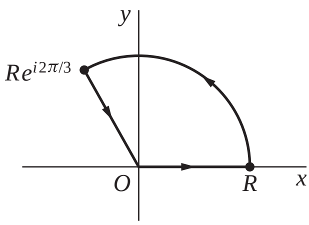

FIGURE 95

9. Let m and n be integers, where $0 \le m < n$. Follow the steps below to derive the integration formula

$$\int_0^\infty \frac{x^{2m}}{x^{2n}+1}\,dx = \frac{\pi}{2n}\csc\left(\frac{2m+1}{2n}\pi\right).$$

(a) Show that the zeros of the polynomial $z^{2n}+1$ lying above the real axis are

$$c_k = \exp\left[i\frac{(2k+1)\pi}{2n}\right] \qquad (k = 0, 1, 2, \ldots, n-1)$$

and that there are none on that axis.

(b) With the aid of Theorem 2 in Sec. 76, show that

$$\operatorname*{Res}_{z=c_k} \frac{z^{2m}}{z^{2n}+1} = -\frac{1}{2n}e^{i(2k+1)\alpha} \qquad (k = 0, 1, 2, \ldots, n-1)$$

where c_k are the zeros found in part *(a)* and

$$\alpha = \frac{2m+1}{2n}\pi.$$

Then use the summation formula

$$\sum_{k=0}^{n-1} z^k = \frac{1-z^n}{1-z} \qquad (z \ne 1)$$

(see Exercise 9, Sec. 8) to obtain the expression

$$2\pi i \sum_{k=0}^{n-1} \operatorname*{Res}_{z=c_k} \frac{z^{2m}}{z^{2n}+1} = \frac{\pi}{n\sin\alpha}.$$

(c) Use the final result in part *(b)* to complete the derivation of the integration formula.

10. The integration formula

$$\int_0^\infty \frac{dx}{[(x^2-a)^2+1]^2} = \frac{\pi}{8\sqrt{2}A^3}[(2a^2+3)\sqrt{A+a} + a\sqrt{A-a}],$$

where a is any real number and $A = \sqrt{a^2+1}$, arises in the theory of case-hardening of steel by means of radio-frequency heating.* Follow the steps below to derive it.

(a) Point out why the four zeros of the polynomial

$$q(z) = (z^2 - a)^2 + 1$$

are the square roots of the numbers $a \pm i$. Then, using the fact that the numbers

$$z_0 = \frac{1}{\sqrt{2}}(\sqrt{A+a} + i\sqrt{A-a})$$

and $-z_0$ are the square roots of $a + i$ (Exercise 5, Sec. 10), verify that $\pm\overline{z_0}$ are the square roots of $a - i$ and hence that z_0 and $-\overline{z_0}$ are the only zeros of $q(z)$ in the upper half plane $\operatorname{Im} z \geq 0$.

(b) Using the method derived in Exercise 7, Sec. 76, and keeping in mind that $z_0^2 = a + i$ for purposes of simplification, show that the point z_0 in part *(a)* is a pole of order 2 of the function $f(z) = 1/[q(z)]^2$ and that the residue B_1 at z_0 can be written

$$B_1 = -\frac{q''(z_0)}{[q'(z_0)]^3} = \frac{a - i(2a^2+3)}{16A^2 z_0}.$$

After observing that $q'(-\overline{z}) = -\overline{q'(z)}$ and $q''(-\overline{z}) = \overline{q''(z)}$, use the same method to show that the point $-\overline{z_0}$ in part *(a)* is also a pole of order 2 of the function $f(z)$, with residue

$$B_2 = \overline{\left\{\frac{q''(z_0)}{[q'(z_0)]^3}\right\}} = -\overline{B_1}.$$

Then obtain the expression

$$B_1 + B_2 = \frac{1}{8A^2 i} \operatorname{Im}\left[\frac{-a + i(2a^2+3)}{z_0}\right]$$

for the sum of these residues.

(c) Refer to part *(a)* and show that $|q(z)| \geq (R - |z_0|)^4$ if $|z| = R$, where $R > |z_0|$. Then, with the aid of the final result in part *(b)*, complete the derivation of the integration formula.

80. IMPROPER INTEGRALS FROM FOURIER ANALYSIS

Residue theory can be useful in evaluating convergent improper integrals of the form

$$(1) \qquad \int_{-\infty}^{\infty} f(x)\sin ax\,dx \quad \text{or} \quad \int_{-\infty}^{\infty} f(x)\cos ax\,dx,$$

*See pp. 359–364 of the book by Brown, Hoyler, and Bierwirth that is listed in Appendix 1.