(5), and hence its left-hand side, tends to zero as R tends to infinity. For the quantity

$$M_R \pi R = \frac{\pi R^2}{(R-\sqrt{2})^2} = \frac{\pi}{\left(1-\dfrac{\sqrt{2}}{R}\right)^2}$$

does not tend to zero. The above theorem does, however, provide the desired limit, namely

$$\lim_{R\to\infty} \int_{C_R} f(z)e^{iz}\,dz = 0,$$

since

$$M_R = \frac{\dfrac{1}{R}}{\left(1-\dfrac{\sqrt{2}}{R}\right)^2} \to 0 \quad \text{as} \quad R \to \infty\,.$$

So it does, indeed, follow from inequality (5) that the left-hand side there tends to zero as R tends to infinity. Consequently, equation (4), together with expression (3) for the residue B_1, tells us that

(6) $$\text{P.V.} \int_{-\infty}^{\infty} \frac{x \sin x\,dx}{x^2+2x+2} = \text{Im}(2\pi i B_1) = \frac{\pi}{e}(\sin 1 + \cos 1).$$

EXERCISES

Use residues to evaluate the improper integrals in Exercises 1 through 8.

1. $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x\,dx}{(x^2+a^2)(x^2+b^2)} \quad (a > b > 0).$

Ans. $\displaystyle\frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right).$

2. $\displaystyle\int_0^{\infty} \frac{\cos ax}{x^2+1}\,dx \quad (a > 0).$

Ans. $\displaystyle\frac{\pi}{2}e^{-a}.$

3. $\displaystyle\int_0^{\infty} \frac{\cos ax}{(x^2+b^2)^2}\,dx \quad (a > 0, b > 0).$

Ans. $\displaystyle\frac{\pi}{4b^3}(1+ab)e^{-ab}.$

4. $\displaystyle\int_0^{\infty} \frac{x \sin 2x}{x^2+3}\,dx.$

Ans. $\displaystyle\frac{\pi}{2}\exp(-2\sqrt{3}).$

5. $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin ax}{x^4+4}\,dx \quad (a > 0).$

Ans. $\dfrac{\pi}{2} e^{-a} \sin a.$

6. $\displaystyle\int_{-\infty}^{\infty} \frac{x^3 \sin ax}{x^4+4}\,dx \quad (a > 0).$

Ans. $\pi e^{-a} \cos a.$

7. $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin x\,dx}{(x^2+1)(x^2+4)}.$

8. $\displaystyle\int_{0}^{\infty} \frac{x^3 \sin x\,dx}{(x^2+1)(x^2+9)}.$

Use residues to find the Cauchy principal values of the improper integrals in Exercises 9 through 11.

9. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x\,dx}{x^2+4x+5}.$

Ans. $-\dfrac{\pi}{e} \sin 2.$

10. $\displaystyle\int_{-\infty}^{\infty} \frac{(x+1)\cos x}{x^2+4x+5}\,dx.$

Ans. $\dfrac{\pi}{e}(\sin 2 - \cos 2).$

11. $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x\,dx}{(x+a)^2+b^2} \quad (b > 0).$

12. Follow the steps below to evaluate the *Fresnel integrals*, which are important in diffraction theory:

$$\int_0^{\infty} \cos(x^2)\,dx = \int_0^{\infty} \sin(x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

(a) By integrating the function $\exp(iz^2)$ around the positively oriented boundary of the sector $0 \le r \le R$, $0 \le \theta \le \pi/4$ (Fig. 99) and appealing to the Cauchy–Goursat theorem, show that

$$\int_0^{R} \cos(x^2)\,dx = \frac{1}{\sqrt{2}}\int_0^{R} e^{-r^2}\,dr - \text{Re}\int_{C_R} e^{iz^2}\,dz$$

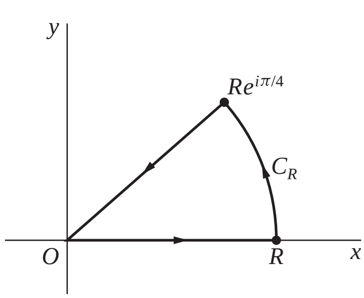

FIGURE 99

and

$$\int_0^R \sin(x^2)\,dx = \frac{1}{\sqrt{2}}\int_0^R e^{-r^2}\,dr - \text{Im}\int_{C_R} e^{iz^2}\,dz,$$

where C_R is the arc $z = Re^{i\theta}$ $(0 \le \theta \le \pi/4)$.

(b) Show that the value of the integral along the arc C_R in part *(a)* tends to zero as R tends to infinity by obtaining the inequality

$$\left|\int_{C_R} e^{iz^2}\,dz\right| \le \frac{R}{2}\int_0^{\pi/2} e^{-R^2\sin\phi}\,d\phi$$

and then referring to the form (2), Sec. 81, of Jordan's inequality.

(c) Use the results in parts *(a)* and *(b)*, together with the known integration formula*

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2},$$

to complete the exercise.

82. INDENTED PATHS

In this and the following section, we illustrate the use of *indented* paths. We begin with an important limit that will be used in the example in this section.

Theorem. *Suppose that*

(a) a function $f(z)$ has a simple pole at a point $z = x_0$ on the real axis, with a Laurent series representation in a punctured disk $0 < |z - x_0| < R_2$ (Fig. 100) *and with residue B_0;*

(b) C_ρ denotes the upper half of a circle $|z - x_0| = \rho$, where $\rho < R_2$ and the clockwise direction is taken.

Then

$$\lim_{\rho \to 0}\int_{C_\rho} f(z)\,dz = -B_0\pi i.$$

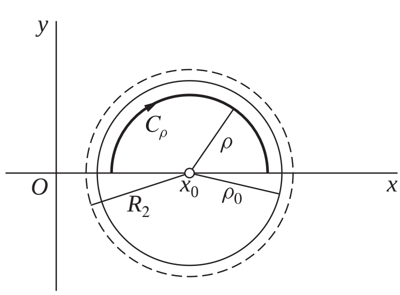

FIGURE 100

*See the footnote with Exercise 4, Sec. 49.