EXERCISES

In Exercises 1 through 4, take the indented contour in Fig. 101 (Sec. 82).

1. Derive the integration formula

$$\int_0^\infty \frac{\cos(ax) - \cos(bx)}{x^2}\, dx = \frac{\pi}{2}(b-a) \qquad (a \ge 0, b \ge 0).$$

Then, with the aid of the trigonometric identity $1 - \cos(2x) = 2\sin^2 x$, point out how it follows that

$$\int_0^\infty \frac{\sin^2 x}{x^2}\, dx = \frac{\pi}{2}.$$

2. Evaluate the improper integral

$$\int_0^\infty \frac{x^a}{(x^2+1)^2}\, dx, \quad \text{where} \quad -1 < a < 3 \text{ and } x^a = \exp(a \ln x).$$

Ans. $\dfrac{(1-a)\pi}{4\cos(a\pi/2)}$.

3. Use the function

$$f(z) = \frac{z^{1/3}\log z}{z^2+1} = \frac{e^{(1/3)\log z}\log z}{z^2+1} \qquad \left(|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right)$$

to derive this pair of integration formulas:

$$\int_0^\infty \frac{\sqrt[3]{x}\ln x}{x^2+1}\, dx = \frac{\pi^2}{6}, \qquad \int_0^\infty \frac{\sqrt[3]{x}}{x^2+1}\, dx = \frac{\pi}{\sqrt{3}}.$$

4. Use the function

$$f(z) = \frac{(\log z)^2}{z^2+1} \qquad \left(|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right)$$

to show that

$$\int_0^\infty \frac{(\ln x)^2}{x^2+1}\, dx = \frac{\pi^3}{8}, \qquad \int_0^\infty \frac{\ln x}{x^2+1}\, dx = 0.$$

Suggestion: The integration formula obtained in Exercise 1, Sec. 79, is needed here.

5. Use the function

$$f(z) = \frac{z^{1/3}}{(z+a)(z+b)} = \frac{e^{(1/3)\log z}}{(z+a)(z+b)} \qquad (|z| > 0, 0 < \arg z < 2\pi)$$

and a closed contour similar to the one in Fig. 103 (Sec. 84) to show formally that

$$\int_0^\infty \frac{\sqrt[3]{x}}{(x+a)(x+b)}\, dx = \frac{2\pi}{\sqrt{3}} \cdot \frac{\sqrt[3]{a} - \sqrt[3]{b}}{a-b} \qquad (a > b > 0).$$

6. Show that

$$\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)} = \frac{\pi}{\sqrt{2}}$$

by integrating an appropriate branch of the multiple-valued function

$$f(z) = \frac{z^{-1/2}}{z^2+1} = \frac{e^{(-1/2)\log z}}{z^2+1}$$

over *(a)* the indented path in Fig. 101, Sec. 82; *(b)* the closed contour in Fig. 103, Sec. 84.

7. The *beta function* is this function of two real variables:

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt \qquad (p>0, q>0).$$

Make the substitution $t = 1/(x+1)$ and use the result obtained in the example in Sec. 84 to show that

$$B(p, 1-p) = \frac{\pi}{\sin(p\pi)} \qquad (0 < p < 1).$$

8. Consider the two simple closed contours shown in Fig. 104 and obtained by dividing into two pieces the annulus formed by the circles C_ρ and C_R in Fig. 103 (Sec. 84). The legs L and $-L$ of those contours are directed line segments along any ray $\arg z = \theta_0$, where $\pi < \theta_0 < 3\pi/2$. Also, Γ_ρ and γ_ρ are the indicated portions of C_ρ, while Γ_R and γ_R make up C_R.

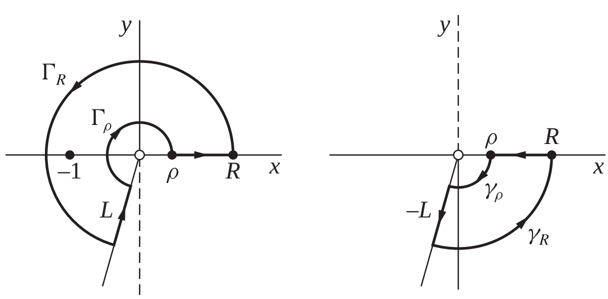

FIGURE 104

(a) Show how it follows from Cauchy's residue theorem that when the branch

$$f_1(z) = \frac{z^{-a}}{z+1} \qquad \left(|z|>0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right)$$

of the multiple-valued function $z^{-a}/(z+1)$ is integrated around the closed contour on the left in Fig. 104,

$$\int_\rho^R \frac{r^{-a}}{r+1}\,dr + \int_{\Gamma_R} f_1(z)\,dz + \int_L f_1(z)\,dz + \int_{\Gamma_\rho} f_1(z)\,dz = 2\pi i \operatorname*{Res}_{z=-1} f_1(z).$$

(b) Apply the Cauchy–Goursat theorem to the branch

$$f_2(z) = \frac{z^{-a}}{z+1} \qquad \left(|z| > 0, \frac{\pi}{2} < \arg z < \frac{5\pi}{2}\right)$$

of $z^{-a}/(z+1)$, integrated around the closed contour on the right in Fig. 104, to show that

$$-\int_{\rho}^{R} \frac{r^{-a}e^{-i2a\pi}}{r+1}\,dr + \int_{\gamma_\rho} f_2(z)\,dz - \int_L f_2(z)\,dz + \int_{\gamma_R} f_2(z)\,dz = 0.$$

(c) Point out why, in the last lines in parts *(a)* and *(b)*, the branches $f_1(z)$ and $f_2(z)$ of $z^{-a}/(z+1)$ can be replaced by the branch

$$f(z) = \frac{z^{-a}}{z+1} \qquad (|z| > 0, 0 < \arg z < 2\pi).$$

Then, by adding corresponding sides of those two lines, derive equation (3), Sec. 84, which was obtained only formally there.

85. DEFINITE INTEGRALS INVOLVING SINES AND COSINES

The method of residues is also useful in evaluating certain definite integrals of the type

$$(1) \qquad \int_0^{2\pi} F(\sin\theta, \cos\theta)\,d\theta.$$

The fact that θ varies from 0 to 2π leads us to consider θ as an argument of a point z on a positively oriented circle C centered at the origin. Taking the radius to be unity, we use the parametric representation

$$(2) \qquad z = e^{i\theta} \qquad (0 \le \theta \le 2\pi)$$

to describe C (Fig. 105). We then refer to the differentation formula (4), Sec. 37, to write

$$\frac{dz}{d\theta} = ie^{i\theta} = iz$$

and recall (Sec. 34) that

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \qquad \text{and} \qquad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

These relations suggest that we make the substitutions

$$(3) \qquad \sin\theta = \frac{z - z^{-1}}{2i}, \qquad \cos\theta = \frac{z + z^{-1}}{2}, \qquad d\theta = \frac{dz}{iz},$$