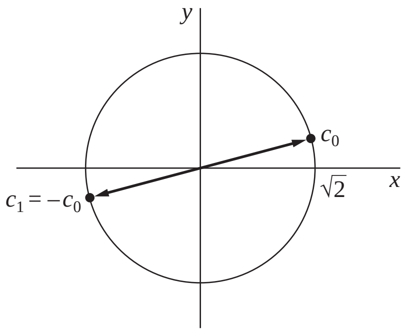

FIGURE 14

$$\cos^2 \frac{\alpha}{2} = \frac{1 + \cos\alpha}{2}, \quad \sin^2 \frac{\alpha}{2} = \frac{1 - \cos\alpha}{2} \tag{4}$$

enable us to write

$$\cos^2 \frac{\pi}{12} = \frac{1}{2}\left(1 + \cos\frac{\pi}{6}\right) = \frac{1}{2}\left(1 + \frac{\sqrt{3}}{2}\right) = \frac{2+\sqrt{3}}{4},$$

$$\sin^2 \frac{\pi}{12} = \frac{1}{2}\left(1 - \cos\frac{\pi}{6}\right) = \frac{1}{2}\left(1 - \frac{\sqrt{3}}{2}\right) = \frac{2-\sqrt{3}}{4}.$$

Consequently,

$$c_0 = \sqrt{2}\left(\sqrt{\frac{2+\sqrt{3}}{4}} + i\sqrt{\frac{2-\sqrt{3}}{4}}\right) = \frac{1}{\sqrt{2}}\left(\sqrt{2+\sqrt{3}} + i\sqrt{2-\sqrt{3}}\right).$$

Since $c_1 = -c_0$, the two square roots of $\sqrt{3} + i$ are, then,

$$\pm\frac{1}{\sqrt{2}}\left(\sqrt{2+\sqrt{3}} + i\sqrt{2-\sqrt{3}}\right). \tag{5}$$

EXERCISES

1. Find the square roots of *(a)* $2i$; *(b)* $1 - \sqrt{3}i$ and express them in rectangular coordinates.

Ans. (a) $\pm\,(1+i)$; *(b)* $\pm\dfrac{\sqrt{3}-i}{\sqrt{2}}$.

2. In each case, find all the roots in rectangular coordinates, exhibit them as vertices of certain squares, and point out which is the principal root:

(a) $(-16)^{1/4}$; *(b)* $(-8 - 8\sqrt{3}i)^{1/4}$.

Ans. (a) $\pm\sqrt{2}(1+i),\ \pm\sqrt{2}(1-i)$; *(b)* $\pm(\sqrt{3}-i),\ \pm(1+\sqrt{3}i)$.

3. In each case, find all the roots in rectangular coordinates, exhibit them as vertices of certain regular polygons, and identify the principal root:

(a) $(-1)^{1/3}$; *(b)* $8^{1/6}$.

Ans. (b) $\pm\sqrt{2},\ \pm\dfrac{1+\sqrt{3}i}{\sqrt{2}},\ \pm\dfrac{1-\sqrt{3}i}{\sqrt{2}}$.

4. According to Sec. 9, the three cube roots of a nonzero complex number z_0 can be written $c_0, c_0\omega_3, c_0\omega_3^2$ where c_0 is the principal cube root of z_0 and

$$\omega_3 = \exp\left(i\frac{2\pi}{3}\right) = \frac{-1+\sqrt{3}i}{2}.$$

Show that if $z_0 = -4\sqrt{2} + 4\sqrt{2}i$, then $c_0 = \sqrt{2}(1+i)$ and the other two cube roots are, in rectangular form, the numbers

$$c_0\omega_3 = \frac{-(\sqrt{3}+1) + (\sqrt{3}-1)i}{\sqrt{2}}, \quad c_0\omega_3^2 = \frac{(\sqrt{3}-1) - (\sqrt{3}+1)i}{\sqrt{2}}.$$

5. *(a)* Let a denote any fixed real number and show that the two square roots of $a+i$ are

$$\pm\sqrt{A}\exp\left(i\frac{\alpha}{2}\right)$$

where $A = \sqrt{a^2+1}$ and $\alpha = \operatorname{Arg}(a+i)$.

(b) With the aid of the trigonometric identities (4) in Example 3 of Sec. 10, show that the square roots obtained in part *(a)* can be written

$$\pm\frac{1}{\sqrt{2}}\left(\sqrt{A+a} + i\sqrt{A-a}\right).$$

(Note that this becomes the final result in Example 3, Sec. 10, when $a = \sqrt{3}$.)

6. Find the four zeros of the polynomial $z^4 + 4$, one of them being

$$z_0 = \sqrt{2}\,e^{i\pi/4} = 1 + i.$$

Then use those zeros to factor $z^2 + 4$ into quadratic factors with real coefficients.

Ans. $(z^2 + 2z + 2)(z^2 - 2z + 2)$.

7. Show that if c is any nth root of unity other than unity itself, then

$$1 + c + c^2 + \cdots + c^{n-1} = 0.$$

Suggestion: Use the first identity in Exercise 9, Sec. 8.

8. *(a)* Prove that the usual formula solves the quadratic equation

$$az^2 + bz + c = 0 \qquad (a \neq 0)$$

when the coefficients a, b, and c are complex numbers. Specifically, by completing the square on the left-hand side, derive the *quadratic formula*

$$z = \frac{-b + (b^2 - 4ac)^{1/2}}{2a},$$

where both square roots are to be considered when $b^2 - 4ac \neq 0$, *(b)* Use the result in part *(a)* to find the roots of the equation $z^2 + 2z + (1 - i) = 0$.

Ans. (b) $\left(-1 + \dfrac{1}{\sqrt{2}}\right) + \dfrac{i}{\sqrt{2}}, \quad \left(-1 - \dfrac{1}{\sqrt{2}}\right) - \dfrac{i}{\sqrt{2}}.$

9. Let $z = re^{i\theta}$ be a nonzero complex number and n a negative integer ($n = -1, -2, \ldots$). Then define $z^{1/n}$ by means of the equation $z^{1/n} = (z^{-1})^{1/m}$ where $m = -n$. By showing that the m values of $(z^{1/m})^{-1}$ and $(z^{-1})^{1/m}$ are the same, verify that $z^{1/n} = (z^{1/m})^{-1}$. (Compare with Exercise 7, Sec. 8.)

11. REGIONS IN THE COMPLEX PLANE

In this section, we are concerned with sets of complex numbers, or points in the z plane, and their closeness to one another. Our basic tool is the concept of an ε *neighborhood*

$$|z - z_0| < \varepsilon \tag{1}$$

of a given point z_0. It consists of all points z lying inside but not on a circle centered at z_0 and with a specified positive radius ε (Fig. 15). When the value of ε is understood or is immaterial in the discussion, the set (1) is often referred to as just a neighborhood. Occasionally, it is convenient to speak of a *deleted neighborhood*, or punctured disk,

$$0 < |z - z_0| < \varepsilon \tag{2}$$

consisting of all points z in an ε neighborhood of z_0 except for the point z_0 itself.

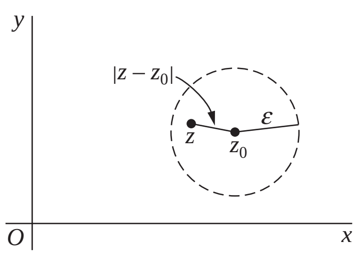

FIGURE 15

A point z_0 is said to be an *interior point* of a set S whenever there is some neighborhood of z_0 that contains only points of S; it is called an *exterior point* of S when there exists a neighborhood of it containing no points of S. If z_0 is neither of these, it is a *boundary point* of S. A boundary point is, therefore, a point all of whose neighborhoods contain at least one point in S and at least one point not in S. The totality of all boundary points is called the *boundary* of S. The circle $|z| = 1$, for instance, is the boundary of each of the sets

$$|z| < 1 \quad \text{and} \quad |z| \leq 1. \tag{3}$$