Note that because $|a| < 1$,

$$|z_2| = \frac{1 + \sqrt{1 - a^2}}{|a|} > 1.$$

Also, since $|z_1 z_2| = 1$, it follows that $|z_1| < 1$. Hence there are no singular points on C, and the only one interior to it is the point z_1. The corresponding residue B_1 is found by writing

$$f(z) = \frac{\phi(z)}{z - z_1} \quad \text{where} \quad \phi(z) = \frac{2/a}{z - z_2}.$$

This shows that z_1 is a simple pole and that

$$B_1 = \phi(z_1) = \frac{2/a}{z_1 - z_2} = \frac{1}{i\sqrt{1 - a^2}}.$$

Consequently,

$$\int_C \frac{2/a}{z^2 + (2i/a)z - 1}\, dz = 2\pi i B_1 = \frac{2\pi}{\sqrt{1 - a^2}};$$

and integration formula (5) follows.

The method just illustrated applies equally well when the arguments of the sine and cosine are integral multiples of θ. One can use equation (2) to write, for example,

$$\cos 2\theta = \frac{e^{i2\theta} + e^{-i2\theta}}{2} = \frac{(e^{i\theta})^2 + (e^{i\theta})^{-2}}{2} = \frac{z^2 + z^{-2}}{2}.$$

EXERCISES

Use residues to evaluate the definite integrals in Exercises 1 through 7.

1. $\displaystyle\int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta}$.

Ans. $\dfrac{2\pi}{3}$.

2. $\displaystyle\int_{-\pi}^{\pi} \frac{d\theta}{1 + \sin^2\theta}$.

Ans. $\sqrt{2}\pi$.

3. $\displaystyle\int_0^{2\pi} \frac{\cos^2 3\theta\, d\theta}{5 - 4\cos 2\theta}$.

Ans. $\dfrac{3\pi}{8}$.

4. $\displaystyle\int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} \quad (-1 < a < 1).$

Ans. $\dfrac{2\pi}{\sqrt{1-a^2}}.$

5. $\displaystyle\int_0^{\pi} \frac{\cos 2\theta\, d\theta}{1 - 2a\cos\theta + a^2} \quad (-1 < a < 1).$

Ans. $\dfrac{a^2\pi}{1-a^2}.$

6. $\displaystyle\int_0^{\pi} \frac{d\theta}{(a + \cos\theta)^2} \quad (a > 1).$

Ans. $\dfrac{a\pi}{\left(\sqrt{a^2-1}\right)^3}.$

7. $\displaystyle\int_0^{\pi} \sin^{2n}\theta\, d\theta \quad (n = 1, 2, \ldots).$

Ans. $\dfrac{(2n)!}{2^{2n}(n!)^2}\pi.$

86. ARGUMENT PRINCIPLE

A function f is said to be *meromorphic* in a domain D if it is analytic throughout D except for poles. Suppose now that f is meromorphic in the domain interior to a positively oriented simple closed contour C and that it is analytic and nonzero on C. The image Γ of C under the transformation $w = f(z)$ is a closed contour, not necessarily simple, in the w plane (Fig. 106). As a point z traverses C in the positive direction, its images w traverses Γ in a particular direction that determines the orientation of Γ. Note that since f has no zeros on C, the contour Γ does not pass through the origin in the w plane.

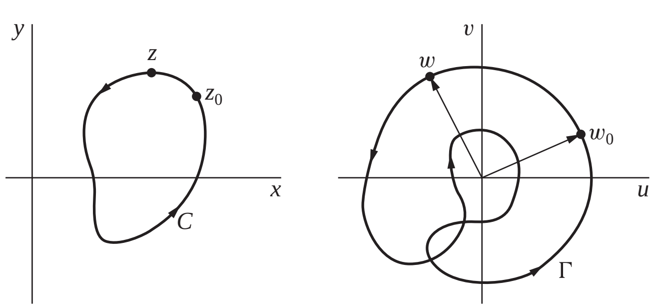

FIGURE 106

Let w_0 and w be points on Γ, where w_0 is fixed and ϕ_0 is a value of $\arg w_0$. Then let $\arg w$ vary continuously, starting with the value ϕ_0, as the point w begins at the point w_0 and traverses Γ once in the direction of orientation assigned to it