and let z be any point on a circle $|z| = R$, where $R > 1$. When such a point is taken, we see that

$$|f(z)| = |a_n|R^n.$$

Also,

$$|g(z)| \le |a_0| + |a_1|R + |a_2|R^2 + \cdots + |a_{n-1}|R^{n-1}.$$

Consequently, since $R > 1$,

$$|g(z)| \le |a_0|R^{n-1} + |a_1|R^{n-1} + |a_2|R^{n-1} + \cdots + |a_{n-1}|R^{n-1};$$

and it follows that

$$\frac{|g(z)|}{|f(z)|} \le \frac{|a_0| + |a_1| + |a_2| + \cdots + |a_{n-1}|}{|a_n|R} < 1$$

if, in addition to being greater than unity,

$$R > \frac{|a_0| + |a_1| + |a_2| + \cdots + |a_{n-1}|}{|a_n|}. \tag{4}$$

That is, $|f(z)| > |g(z)|$ when $R > 1$ and inequality (4) is satisfied. Rouché's theorem then tells us that $f(z)$ and $f(z) + g(z)$ have the same number of zeros, namely n, inside C. Hence we may conclude that $P(z)$ has precisely n zeros, counting multiplicities, in the plane.

Note how Liouville's theorem in Sec. 53 only ensured the existence of at least one zero of a polynomial; but Rouché's theorem actually ensures the existence of n zeros, counting multiplicities.

EXERCISES

1. Let C denote the unit circle $|z| = 1$, described in the positive sense. Use the theorem in Sec. 86 to determine the value of $\Delta_C \arg f(z)$ when

(a) $f(z) = z^2$; *(b)* $f(z) = (z^3 + 2)/z$; *(c)* $f(z) = (2z - 1)^7/z^3$.

Ans. *(a)* 4π; *(b)* -2π; *(c)* 8π.

2. Let f be a function which is analytic inside and on a positively oriented simple closed contour C, and suppose that $f(z)$ is never zero on C. Let the image of C under the transformation $w = f(z)$ be the closed contour Γ shown in Fig. 107. Determine the value of $\Delta_C \arg f(z)$ from that figure; and, with the aid of the theorem in Sec. 86, determine the number of zeros, counting multiplicities, of f interior to C.

Ans. 6π; 3.

3. Using the notation in Sec. 86, suppose that Γ does not enclose the origin $w = 0$ and that there is a ray from that point which does not intersect Γ. By observing that the

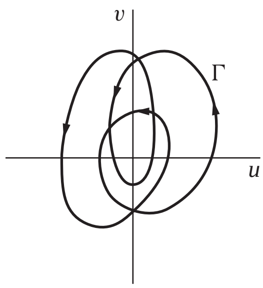

FIGURE 107

absolute value of $\Delta_C \arg f(z)$ must be less than 2π when a point z makes one cycle around C and recalling that $\Delta_C \arg f(z)$ is an integral multiple of 2π, point out why the winding number of Γ with respect to the origin $w = 0$ must be zero.

4. Suppose that a function f is meromorphic in the domain D interior to a simple closed contour C on which f is analytic and nonzero, and let D_0 denote the domain consisting of all points in D except for poles. Point out how it follows from the lemma in Sec. 27 and Exercise 10, Sec. 76, that if $f(z)$ is not identically equal to zero in D_0, then the zeros of f in D are all of finite order and that they are finite in number.

Suggestion: Note that if a point z_0 in D is a zero of f that is not of finite order, then there must be a neighborhood of z_0 throughout which $f(z)$ is identically equal to zero.

5. Suppose that a function f is analytic inside and on a positively oriented simple closed contour C and that it has no zeros on C. Show that if f has n zeros z_k $(k = 1, 2, \ldots, n)$ inside C, where each z_k is of multiplicity m_k, then

$$\int_C \frac{zf'(z)}{f(z)}\, dz = 2\pi i \sum_{k=1}^{n} m_k z_k.$$

[Compare with equation (8), Sec. 86, when $P = 0$ there.]

6. Determine the number of zeros, counting multiplicities, of the polynomial

(a) $z^6 - 5z^4 + z^3 - 2z$; *(b)* $2z^4 - 2z^3 + 2z^2 - 2z + 9$

inside the circle $|z| = 1$.

Ans. *(a)* 4; *(b)* 0.

7. Determine the number of zeros, counting multiplicities, of the polynomial

(a) $z^4 + 3z^3 + 6$; *(b)* $z^4 - 2z^3 + 9z^2 + z - 1$; *(c)* $z^5 + 3z^3 + z^2 + 1$

inside the circle $|z| = 2$.

Ans. *(a)* 3; *(b)* 2; *(c)* 5.

8. Determine the number of roots, counting multiplicities, of the equation

$$2z^5 - 6z^2 + z + 1 = 0$$

in the annulus $1 \le |z| < 2$.

Ans. 3.

9. Show that if c is a complex number such that $|c| > e$, then the equation $cz^n = e^z$ has n roots, counting multiplicities, inside the circle $|z| = 1$.

10. Let two functions f and g be as in the statement of Rouché's theorem in Sec. 87, and let the orientation of the contour C there be positive. Then define the function

$$\Phi(t) = \frac{1}{2\pi i}\int_C \frac{f'(z) + tg'(z)}{f(z) + tg(z)}\,dz \qquad (0 \le t \le 1)$$

and follow these steps below to give another proof of Rouché's theorem.

(a) Point out why the denominator in the integrand of the integral defining $\Phi(t)$ is never zero on C. This ensures the existence of the integral.

(b) Let t and t_0 be any two points in the interval $0 \le t \le 1$ and show that

$$|\Phi(t) - \Phi(t_0)| = \frac{|t - t_0|}{2\pi}\left|\int_C \frac{fg' - f'g}{(f + tg)(f + t_0 g)}\,dz\right|.$$

Then, after pointing out why

$$\left|\frac{fg' - f'g}{(f + tg)(f + t_0 g)}\right| \le \frac{|fg' - f'g|}{(|f| - |g|)^2}$$

at points on C, show that there is a positive constant A, which is independent of t and t_0, such that

$$|\Phi(t) - \Phi(t_0)| \le A|t - t_0|.$$

Conclude from this inequality that $\Phi(t)$ is continuous on the interval $0 \le t \le 1$.

(c) By referring to equation (8), Sec. 86, state why the value of the function Φ is, for each value of t in the interval $0 \le t \le 1$, an integer representing the number of zeros of $f(z) + tg(z)$ inside C. Then conclude from the fact that Φ is continuous, as shown in part *(b)*, that $f(z)$ and $f(z) + g(z)$ have the same number of zeros, counting multiplicities, inside C.

88. INVERSE LAPLACE TRANSFORMS

Suppose that a function F of the complex variable s is analytic throughout the finite s plane except for a finite number of isolated singularities. Then let L_R denote a vertical line segment from $s = \gamma - iR$ to $s = \gamma + iR$, where the constant γ is positive and large enough that the singularities of F all lie to the left of that segment (Fig. 108). A new function f of the real variable t is defined for positive values of t by means of the equation

$$f(t) = \frac{1}{2\pi i}\lim_{R\to\infty}\int_{L_R} e^{st}F(s)\,ds \qquad (t > 0), \tag{1}$$

provided this limit exists. Expression (1) is usually written

$$f(t) = \frac{1}{2\pi i}\ \text{P.V.}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st}F(s)\,ds \qquad (t > 0) \tag{2}$$